The Serret-Andoyer Formalism in Rigid-Body Dynamics: I. Symmetries and Perturbations

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Abstract

This paper reviews the Serret-Andoyer (SA) canonical formalism in rigid-body dynamics, and presents some new results. As well known, the problem of unsupported and unperturbed rigid rotator can be reduced. The availability of this reduction is offered by the underlying symmetry, which stems from conservation of the angular momentum and rotational kinetic energy. When a perturbation is turned on, these quantities are no longer preserved. Nonetheless, the language of reduced description remains extremely instrumental even in the perturbed case. We describe the canonical reduction performed by the Serret-Andoyer (SA) method, and discuss its applications to attitude dynamics and to the theory of planetary rotation. Specifically, we consider the case of angular-velocity-dependent torques, and discuss the variation-of-parameters inherent antinomy between canonicity and osculation. Finally, we address the transformation of the Andoyer variables into action-angle ones, using the method of Sadov.

1 INTRODUCTION

One of the classical problems of mechanics is that of a free motion of a rigid body, usually referred to as the *Euler-Poinsot* problem. The formulation and solution of this problem are usually performed in two steps. First, the *dynamics* of the rotation are represented by differential equations for the components of the body angular velocity. As well known, these equations admit a closed-form solution in terms of Jacobi's elliptic functions [1]. Second, the *kinematic* equations are utilised to transform the body angular velocity into a spatial inertial frame. While this classical formulation is widespread among engineers, astronomers exhibit a marked preference for formulation that takes advantage of the internal symmetries of the Euler-Poinsot setting [2]. The existence of internal symmetries indicates that the unperturbed Euler-Poinsot problem can be reduced to a smaller number of variables, whereafter the disturbed setting can be treated as a perturbation expressed through those new variables (disturbance may be called into being by various reasons - physical torques; inertial torques emerging in non-inertial frames of reference; non-rigidity of the rotator).

There exist reasons for performing this reduction in the Hamiltonian form and making the resulting reduced variables canonical. The first reason is that a perturbed Hamiltonian system can be analytically solved in any order over the parameter entering the perturbation.[1] A celebrated example of a

[1] While in the past such analytical solutions were built by means of the von Zeipel method, more efficient is the procedure independently offered by Hori [3] and Deprit [4]. An explanation of the method can be found in [5]. A concise introduction into the subject can be found in Chapter 8 of the second volume of [6] and in Chapter 5 of [7].

successful application of the Hori-Deprit method to a geophysical problem is given by the theory of rigid Earth rotation offered by Kinoshita [8] and further developed by joint efforts of Escapa, Getino, & Ferrándiz [9, 10], Getino & Ferrándiz [11], and Kinoshita & Souchay [12].

The second advantage of the canonical description originates from the convenience of numerical implementation: symplectic schemes are well-known for their good stability and precision. (See, for example, Yoshida [13].) This is why the Hamiltonian methods are especially beneficial at long time scales, feature important in astronomy. (Laskar & Robutel [14]; Touma & Wisdom [15, 16])

Several slightly different sets of canonical variables are used for modelling rigid-body dynamics and kinematics. Most popular is the set suggested in 1923 by Andoyer [17]. This set is not completely reduced: while three of its elements are constants (in the unperturbed free-spin case), three others are permitted to evolve. Andoyer arrived to his variables through an exercise in spherical trigonometry. Starting from the Eulerian angles, he performed a change of variables, change that depended upon the values of the angular-momentum components. Canonicity of this change of variables should then be proven by a direct construction of the corresponding generating function. Much later, the study by Andoyer was amended by Deprit [18] who demonstrated that the canonicity of Andoyer's transformation may be proven by using differential forms and without resorting to explicitly finding a generating function.

A full reduction of the Euler-Poinsot problem, that has the Andoyer variables as its starting point, was recently offered by Deprit & Elipe [2]. It would be interesting to notice that, historically, the pioneer canonical treatment of the problem, too, was formulated in a completely reduced way, i.e., in terms of canonically-conjugate constants of motion. These constants, presently known as Serret elements, are named after the 19th century French mathematician Joseph Alfred Serret who discovered these variables by solving the Hamilton-Jacobi equation written in terms of the Eulerian coordinates [19]. Serret's treatment was later simplified by Radeau [20] and Tisserand [21]. However, for the first time the Serret elements appeared in an earlier publication by Richelot [22].

Probably the most distinguishing feature of the canonical approach is that it permits reduction of the torque-free rotational dynamics to a one-and-a-half degrees of freedom. In essence, such a formulation reduces the dynamics by capturing the underlying symmetry of the free rigid body problem – the symmetry taking its origin from the conservation of energy and angular momentum. As a result, the entire dynamics can be expressed by differential equations for two of the Eulerian angles and one of the conjugate momenta – equations that are readily integrable by quadrature. The corresponding one-and-a-half-degrees-of-freedom Hamiltonian yields a phase portrait, which is similar to that of the simple pendulum and contains a separatrix confining the librational motions [18].

In the canonical formulation of attitude-mechanics problems, incorporation of perturbation and/or control torques into the picture is a subtle operation. In particular, when the perturbing torque is angular-velocity-dependent, the canonicity demand comes into a contradiction with the osculation condition. In other words, the expression for the angular velocity through the osculating Serret or Andoyer variables acquires a correction called the *convective* term.

To better understand the latter observation, we shall use an orbital-dynamics analogy. This shall be convenient, because both orbital and rotational dynamics employ the method of variation-of-parameters to model perturbing inputs. In both cases, the coordinates (Cartesian, in the orbital case, or Eulerian in the rotational case) are expressed, in a non-perturbed setting, via the time and six adjustable constants called elements (orbital elements or rotational elements, accordingly). If, under disturbance, we use these expressions as an ansatz and endow the "constants" with time dependence, then the perturbed velocity (orbital or angular) will consist of a partial derivative with respect to the time, plus the convective term, one that includes the time derivatives of the variable "constants." Out of sheer convenience, the so-called Lagrange constraint is often imposed. This constraint nullifies the convective term and, thereby, guarantees that the functional dependence of the velocity upon the time and "constants" stays, under perturbations, the same as it used to be in the undisturbed setting. The variable "constants" obeying this condition are called osculating elements. Otherwise, they are simply called orbital elements (in orbital mechanics) or rotational elements (in attitude mechanics).

When the dynamical equations, written in terms of the "constants," are demanded to be canonical, these "constants" are the Delaunay elements, in the orbital case, or the initial values of the Andoyer elements, in the spin case. These two sets of elements share a feature not readily apparent: in certain cases, the standard equations render these elements non-osculating.

In attitude dynamics, the Andoyer variables come out non-osculating when the perturbation depends upon the angular velocity. For example, since a transition to a non-inertial frame is an angular-velocity-dependent perturbation, then amendment of the dynamical equations by only adding extra terms to the Hamiltonian makes these equations render non-osculating Andoyer variables. To make them osculating, extra terms must be added in the dynamical equations (and then these equations will no longer be symplectic). Calculations in terms of non-osculating variables are mathematically valid, but their physical interpretation is not always easy.

The purpose of this paper will be threefold. We shall first aim to provide the reader with a coherent review of the Serret-Andoyer (SA) formalism; second, we shall dwell on expressing angular-velocity-dependent disturbing torques via the Andoyer variables; and third, we shall consider introduction and physical interpretation of the Andoyer variables in precessing reference frames, with applications to planetary rotation. A subsequent publication [50] will deal with the Andoyer variables geometry based on the theory of Hamiltonian systems on Lie groups and will fill the existing gap in the control literature by using the SA modelling of rigid-body dynamics in order to derive nonlinear asymptotically stabilising controllers.

2 THE EULERIAN VARIABLES

Through this entire section the compact notations $s_x = \sin(x)$, $c_x = \cos(x)$ will be used. The angular-velocity and angular-momentum vectors, as seen in the principal axes of the body, will be denoted by low-case letters: $\boldsymbol{\omega}$ and $\mathbf{g}$. The same two vectors in an inertial coordinate system will be denoted with capital letters: $\boldsymbol{\Omega}$ and $\mathbf{G}$.

2.1 Basic formulae

Consider a rotation of a rigid body about its center of mass, O. The body frame, $\mathscr{B}$, is a Cartesian dextral frame that is centered at the point O and is defined by the unit vectors $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, constituting the fundamental plane, and by $\hat{\mathbf{b}}_3 = \hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2$. The attitude of $\mathscr{B}$ will be studied relative to an inertial Cartesian dextral frame, $\mathscr{I}$, defined by the unit vectors $\hat{\mathbf{s}}_1$, $\hat{\mathbf{s}}_2$ lying on the fundamental plane, and by $\hat{\mathbf{s}}_3 = \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2$. These frames are depicted in Fig. 2.

A transformation from $\mathscr{I}$ to $\mathscr{B}$ may be implemented by three consecutive rotations making a $3-1-3$ sequence. Let the line of nodes (LON) OQ_2 be the intersection of the body fundamental plane and the inertial fundamental plane, as shown in Fig. 2. Let $\hat{\mathbf{l}}$ be a unit vector pointing along OQ_2. The rotation sequence can now be defined as follows:[2]

- $R(\phi, \hat{\mathbf{s}}_3)$, a rotation about $\hat{\mathbf{s}}_3$ by $0 \leq \phi \leq 2\pi$, mapping $\hat{\mathbf{s}}_1$ onto $\hat{\mathbf{l}}$;
- $R(\theta, \hat{\mathbf{l}})$, a rotation about $\hat{\mathbf{l}}$ by $0 \leq \theta \leq \pi$, mapping $\hat{\mathbf{s}}_3$ onto $\hat{\mathbf{b}}_3$;
- $R(\psi, \hat{\mathbf{b}}_3)$, a rotation about $\hat{\mathbf{b}}_3$ by $0 \leq \psi \leq 2\pi$, mapping $\hat{\mathbf{l}}$ onto $\hat{\mathbf{b}}_1$.

The composite rotation, $R \in SO(3)$, transforming any inertial vector into the body frame, is given by

$$R(\phi, \theta, \psi) = R(\psi, \hat{\mathbf{b}}_3) R(\theta, \hat{\mathbf{l}}) R(\phi, \hat{\mathbf{s}}_3) \tag{1}$$

Evaluation of this product gives

$$R(\phi, \theta, \psi) = \begin{bmatrix} \mathrm{c}_\psi \mathrm{c}_\phi - \mathrm{s}_\psi \mathrm{c}_\theta \mathrm{s}_\phi & \mathrm{c}_\psi \mathrm{s}_\phi + \mathrm{s}_\psi \mathrm{c}_\theta \mathrm{c}_\phi & \mathrm{s}_\psi \mathrm{s}_\theta \\ -\mathrm{s}_\psi \mathrm{c}_\phi - \mathrm{c}_\psi \mathrm{c}_\theta \mathrm{s}_\phi & -\mathrm{s}_\psi \mathrm{s}_\phi + \mathrm{c}_\psi \mathrm{c}_\theta \mathrm{c}_\phi & \mathrm{c}_\psi \mathrm{s}_\theta \\ \mathrm{s}_\theta \mathrm{s}_\phi & -\mathrm{s}_\theta \mathrm{c}_\phi & \mathrm{c}_\theta \end{bmatrix} . \tag{2}$$

To write the kinematic equations, we recall that the *body angular-velocity* vector, $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$ satisfies [24]

$$\widehat{\boldsymbol{\omega}} = -\dot{R} R^T \tag{3}$$

where the hat map $\widehat{(\cdot)} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is the usual Lie algebra isomorphism and

$$\widehat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} . \tag{4}$$

Insertion of Eq. (2) into (3) yields the well-known expressions for the components of the vector of the body angular velocity $\boldsymbol{\omega}$:

$$\omega_1 = \dot{\phi} \mathrm{s}_\theta \mathrm{s}_\psi + \dot{\theta} \mathrm{c}_\psi \quad , \tag{5}$$
$$\omega_2 = \dot{\phi} \mathrm{c}_\psi \mathrm{s}_\theta - \dot{\theta} \mathrm{s}_\psi \quad , \tag{6}$$
$$\omega_3 = \dot{\psi} + \dot{\phi} \mathrm{c}_\theta \quad , \tag{7}$$

[2] Be mindful that in the physics and engineering literature the Euler angles are traditionally denoted with (ϕ, θ, ψ). In the literature on the Earth rotation, the inverse convention, (ψ, θ, ϕ), is in use. [8, 9, 10, 11, 12]

while the action of the rotation matrix (2) upon the body angular velocity gives the inertial-frame-related (sometimes called spatial) angular velocity:

$$\boldsymbol{\Omega} \; = \; R^T \, \boldsymbol{\omega} \quad , \tag{8}$$

with the following components:

$$\Omega_1 \; = \; \dot{\theta} \; c_\phi \; + \; \dot{\psi} \; s_\theta \; s_\phi \quad , \tag{9}$$

$$\Omega_2 \; = \; \dot{\theta} \; s_\phi \; + \; \dot{\psi} \; s_\theta \; c_\phi \quad , \tag{10}$$

$$\Omega_3 \; = \; \dot{\phi} \; + \; \dot{\psi} \; c_\theta \quad . \tag{11}$$

In the body frame, attitude dynamics are usually formulated by means of the *Euler-Poinsot* equations. In a free-spin case, these equations look as

$$\mathbb{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbb{I}\boldsymbol{\omega} = 0 \quad , \tag{12}$$

$\mathbb{I}$ being the inertia tensor. To arrive to these equations, one has to start out with the Lagrangian,

$$\mathcal{L}(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega} \quad , \tag{13}$$

to substitute therein expressions (5 - 7); to write down the appropriate Euler-Lagrange variational equations for the Euler angles; and then to use the expressions (5 - 7) again, in order to rewrite these Euler-Lagrange equations in terms of $\boldsymbol{\omega}$. This will result in (12). The same sequence of operations carried out on a perturbed Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ will produce the forced Euler-Poinsot equations:

$$\mathbb{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbb{I}\boldsymbol{\omega} = \mathbf{u} \quad , \tag{14}$$

$\mathbf{u} \in \mathbb{R}^3$ being the body-frame-related torque that can be expressed through derivatives of $\Delta\mathcal{L}$. The Euler-Poinsot description of the motion is essentially Lagrangian. Alternatively, the equations of attitude dynamics can be cast into a Hamiltonian shape.

A most trivial but important observation can be made simply from looking at (14). If one rewrites (14) not in terms of velocities but in terms of the Euler angles, three differential equations of the second order will emerge. Their solution will depend on the time and six adjustable constants. Hence, no matter which description one employs – Lagrangian or Hamiltonian – the number of emerging integration constants will always be six.

2.2 Hamiltonian description. The free-spin case.

From here forth we shall assume that the body axes coincide with the principal axes of inertia, so we can write

$$\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3). \tag{15}$$

Having substituted expressions (5 - 7) and (15) into (13), one can easily write the generalised momenta $p_n = \Phi\,,\ \Theta\,,\ \Psi$ conjugate to the configuration variables $q_n = \phi\,,\ \theta\,,\ \psi$ [2, 28]:

$$\Phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = I_1 \mathrm{s}_\theta \mathrm{s}_\psi(\dot\phi \mathrm{s}_\psi \mathrm{s}_\theta + \dot\theta \mathrm{c}_\psi) + I_2 \mathrm{s}_\theta \mathrm{c}_\psi(\dot\phi \mathrm{c}_\psi \mathrm{s}_\theta - \dot\theta \mathrm{s}_\psi) + I_3 \mathrm{c}_\theta(\dot\phi \mathrm{c}\theta + \dot\psi) \quad , \tag{16}$$

$$\Theta = \frac{\partial\mathcal{L}}{\partial\dot\theta} = I_1 \mathrm{c}_\psi(\dot\phi \mathrm{s}_\theta \mathrm{s}_\psi + \dot\theta \mathrm{c}_\psi) - I_2 \mathrm{s}_\psi(\dot\phi \mathrm{c}_\phi \mathrm{s}_\theta - \dot\theta \mathrm{s}_\psi) \quad , \tag{17}$$

$$\Psi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = I_3(\dot\phi \mathrm{c}_\theta + \dot\psi) \quad . \tag{18}$$

The inverse relations will assume the form of

$$\dot\phi = -\frac{I_1 \mathrm{c}_\psi(\Psi \mathrm{c}_\theta \mathrm{c}_\psi - \Phi \mathrm{c}_\psi + \Theta \mathrm{s}_\theta \mathrm{s}_\psi) + I_2 \mathrm{s}_\psi(\Psi \mathrm{c}_\theta \mathrm{s}_\psi - \Phi \mathrm{s}_\psi - \Theta \mathrm{c}_\psi \mathrm{s}_\theta)}{I_1 I_2 \mathrm{s}_\theta^2} \quad , \tag{19}$$

$$\dot\theta = \frac{I_2 \mathrm{c}_\psi(\Phi \mathrm{s}_\psi + \Theta \mathrm{s}_\theta \mathrm{c}_\psi - \Psi \mathrm{s}_\psi \mathrm{c}_\theta) + I_1 \mathrm{s}_\psi(\Psi \mathrm{c}_\psi \mathrm{c}_\theta - \Phi \mathrm{c}_\psi + \Theta \mathrm{s}_\theta \mathrm{s}_\psi)}{I_1 I_2 \mathrm{s}_\theta} \quad , \tag{20}$$

$$\begin{aligned}\dot\psi &= -\frac{I_1 I_3 \mathrm{c}_\psi(\Phi \mathrm{c}_\theta \mathrm{c}_\psi - \Psi \mathrm{c}_\psi \mathrm{c}_\theta^2 - \Theta \mathrm{s}_\psi \mathrm{s}_\theta \mathrm{c}_\theta) + I_3 I_2 \mathrm{s}_\psi(\Phi \mathrm{c}_\theta \mathrm{s}_\psi - \Psi \mathrm{c}_\theta^2 \mathrm{s}_\psi + \Theta \mathrm{c}_\psi \mathrm{s}_\theta \mathrm{c}_\theta)}{I_1 I_2 I_3 \mathrm{s}_\theta^2} \\ &+ \frac{\Psi}{I_3}\end{aligned} \tag{21}$$

Substitution of (19 - 21) into the Legendre-transformation formula

$$\mathcal{H} \ = \ \Phi\,\dot\phi \ + \ \Theta\,\dot\theta \ + \ \Psi\,\dot\psi \ - \ \mathcal{L} \tag{22}$$

will then readily give us the free-spin Hamiltonian:

$$\begin{aligned}\mathcal{H}(\phi,\theta,\psi,\Phi,\Theta,\Psi) \ &= \ \frac{1}{2}\left(\frac{\mathrm{s}_\psi^2}{I_1} + \frac{\mathrm{c}_\psi^2}{I_2}\right)\left(\frac{\Phi - \Psi \mathrm{c}_\theta}{\mathrm{s}_\theta}\right)^2 + \frac{\Psi^2}{2I_3} + \frac{1}{2}\left(\frac{\mathrm{c}_\psi^2}{I_1} + \frac{\mathrm{s}_\psi^2}{I_2}\right)\Theta^2 \\ &+ \ \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\left(\frac{\Phi - \Psi\ \mathrm{c}_\theta}{\mathrm{s}_\theta}\right)\Theta\ \mathrm{s}_\psi\ \mathrm{c}_\psi.\end{aligned} \tag{23}$$

In the above Hamiltonian the coordinate ϕ (the angle of rotation about the inertial axis $\mathbf{s}_3$) is cyclic (ignorable), so that the appropriate momentum Φ is an integral of motion. This symmetry implies a possibility of reduction of the free-spin problem to only two degrees of freedom. Based on this observation, Serret [19] raised the following question: Is there a canonical transformation capable of reducing the amount of degrees of freedom even further? We shall discuss this issue in the following section. To get this program accomplished, we shall need to know the relationship between the body rotational angular momentum and the conjugate momenta. Let $\mathbf{g} = \sum g_i \mathbf{b}_i$ and $\mathbf{G} = \sum G_i \mathbf{s}_i$ be the angular momentum in the body frame and in inertial frame, accordingly. By plugging (5 - 7) into

$$\mathbf{g} = \mathbb{I}\boldsymbol{\omega} \tag{24}$$

and by subsequent insertion of (19 - 21) therein, one easily arrives at

$$\begin{aligned} g_1 &= \frac{\Phi \mathrm{s}_\psi + \Theta \mathrm{s}_\theta \mathrm{c}_\psi - \Psi \mathrm{s}_\psi \mathrm{c}_\theta}{\mathrm{s}_\theta} \quad , & (25) \\ g_2 &= \frac{\Phi \mathrm{c}_\psi - \Theta \mathrm{s}_\psi \mathrm{s}_\theta - \Psi \mathrm{c}_\psi \mathrm{c}_\theta}{\mathrm{s}_\theta} \quad , & (26) \\ g_3 &= \Psi \quad . & (27) \end{aligned}$$

Notice the symplectic structure defined by the components of the angular momentum: It is a matter of computing partial derivatives to check that the Poisson brackets are

$$(g_1, g_2) = -g_3, \qquad (g_2, g_3) = -g_1, \qquad (g_3, g_1) = -g_2 \tag{28}$$

By the same token, insertion of expressions (9 - 11) into

$$\mathbf{G} = \mathbb{I}\boldsymbol{\Omega} \quad , \tag{29}$$

with the subsequent use of (19 - 21), entails:

$$\begin{aligned} G_1 &= \frac{\Psi \mathrm{s}_\phi + \Theta \mathrm{s}_\theta \mathrm{c}_\phi - \Phi \mathrm{s}_\phi \mathrm{c}_\theta}{\mathrm{s}_\theta} \quad , & (30) \\ G_2 &= \frac{\Phi \mathrm{c}_\theta \mathrm{c}_\phi - \Psi \mathrm{c}_\phi + \Theta \mathrm{s}_\phi \mathrm{s}_\theta}{\mathrm{s}_\theta} \quad , & (31) \\ G_3 &= \Phi \quad , & (32) \end{aligned}$$

yielding the symplectic structure

$$(G_1, G_2) = G_3, \qquad (G_2, G_3) = G_1, \qquad (G_3, G_1) = G_2. \tag{33}$$

from the symplectic structure (28) and the expression (50) of the Hamiltonian, there results

$$\dot{g}_1 \;=\; (g_1, \mathcal{H}) = -\left(\frac{1}{I_2} - \frac{1}{I_3}\right) g_2 g_3, \tag{34}$$

$$\dot{g}_2 \;=\; (g_2, \mathcal{H}) = -\left(\frac{1}{I_3} - \frac{1}{I_1}\right) g_3 g_1, \tag{35}$$

$$\dot{g}_1 \;=\; (g_2, \mathcal{H}) = -\left(\frac{1}{I_1} - \frac{1}{I_1}\right) g_1 g_2. \tag{36}$$

This system is integrable because it admits two integrals, the energy (13) and the norm of the angular momentum $|\mathbf{G}| = G$. With this integral, we may regard the phase space of (34)-(36) as a foliation of invariants manifolds

$$S^2(G) = \{(g_1, g_2, g_3) | g_1^2 + g_2^2 + g_3^2 = G^2\}. \tag{37}$$

The trajectories will be the level contours of the *energy ellipsoid*,

$$T_{kin} = \frac{1}{2}\,\boldsymbol{\omega} \cdot \mathbf{G} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = \frac{1}{2}\left(\frac{g_1^2}{I_1} + \frac{g_2^2}{I_2} + \frac{g_3^2}{I_3}\right), \tag{38}$$

on the sphere (37), as can be seen in Figure 1.

Below we shall need also the expressions for the canonical momenta via the components of the angular momentum $\mathbf{G}$. As agreed above, the Euler angles ϕ, θ, ψ determine the orientation of the body relative to some inertial reference frame. Let now the angles ϕ_o, J, l define the orientation of the body relative to the invariable plane (one orthogonal to the angular-momentum vector $\mathbf{G}$), as in Fig. 2; and let h, I, $g - \phi_o$ be the Euler angles defining the orientation of the invariable plane relative to the reference one.[3] Evidently,

$$g_1 \;=\; G\, s_J\, s_l \quad , \qquad g_2 \;=\; G\, s_J\, c_l \quad , \qquad g_3 \;=\; G\, c_J \quad . \tag{39}$$

[3] It would be natural to denote the orientation of the body relative to the invariable plane with ϕ_o, θ_o, ψ_o, but we prefer to follow the already established notations.

It is now straightforward from (25 - 27) and (39) that

$$\Phi \;=\; g_1\, s_\theta\, s_\psi \,+\, g_2\, s_\theta\, c_\psi \,+\, g_3\, c_\theta \;=\; G\, \left(c_\theta\, c_J \,+\, s_\theta\, s_J\, c_{(\psi - l)} \right) \;=\; G\, c_I \quad , \tag{40}$$

$$\Theta \;=\; g_1\, c_\psi \,-\, g_2\, s_\psi \;=\; G\, s_J\, s_{(l\,-\,\psi)} \quad , \tag{41}$$

$$\Psi \;=\; g_3 \;=\; G\, c_J \quad . \tag{42}$$

3 THE SERRET-ANDOYER TRANSFORMATION

3.1 Richelot (1850), Serret (1866), Radau (1869), Tisserand (1889)

The method presently referred to as the Hamilton-Jacobi one is based on the Jacobi equation derived circa 1840. Though Jacobi's book [29] was published only in 1866, the equation became known to the scientific community already in 1842 when Jacobi included it into his lecture course. In 1850 Richelot [22] suggested six constants of motion grouped into three canonically-conjugate pairs. These constants became the attitude-dynamics analogues of the Delaunay variables emerging in the theory of orbits. Later Serret [19] wrote down the explicit form for the generating function responsible for the canonical transformation from $(\psi, \theta, \phi, \Psi, \Theta, \Phi)$ to Richelot's constants. His treatment was further polished by Radau [20] and explained in detail by Tisserand [21]

The canonical transformation, undertaken by Serret,

$$(\; q_1 \;\equiv\; \phi\, ,\; q_2 \;\equiv\; \theta\, ,\; q_3 \;\equiv\; \psi\, ,\; p_1 \;\equiv\; \Phi\, ,\; p_2 \;\equiv\; \Theta\, ,\; p_3 \;\equiv\; \Psi\; ;\;\; \mathcal{H}(q,\, p)\;) \;\;\rightarrow$$

$$(\; Q_1\, ,\; Q_2\, ,\; Q_3\, ,\; P_1\, ,\; P_2\, ,\; P_3\; ;\;\; \mathcal{H}^*(Q,\, P)\;) \quad , \tag{43}$$

is based on the fact well known to the mathematicians of the XIXth century: since both sets, $(\,\mathbf{q},\, \mathbf{p},\; \mathcal{H}(\mathbf{q}, \mathbf{p})\,)$ and $(\,\mathbf{Q},\, \mathbf{P},\, \mathcal{H}^*(\mathbf{Q}, \mathbf{P})\,)$, are postulated to satisfy the Hamiltonian equations, then the infinitesimally small quantities

$$d\zeta \;=\; \mathbf{p}^T\, d\mathbf{q} \;-\; \mathcal{H}\, dt \tag{44}$$

and

$$d\tilde{\zeta} \;=\; \mathbf{Q}^T\, d\mathbf{P} \;+\; \mathcal{H}^*\, dt \quad , \tag{45}$$

are perfect differentials, and so is their sum

$$dS \;\equiv\; d\zeta \;+\; d\tilde{\zeta} = \;\mathbf{p}^T\, d\mathbf{q} \;+\; \mathbf{Q}^T\, d\mathbf{P} \;-\; \left(\mathcal{H} \;-\; \mathcal{H}^*\right)\, dt \quad . \tag{46}$$

If we start with a system described with $(\,\mathbf{q},\, \mathbf{p},\, \mathcal{H}(\mathbf{q}, \mathbf{p})\,)$, it is worth looking for such a re-parameterisation $(\,\mathbf{Q},\, \mathbf{P},\, \mathcal{H}^*(\mathbf{Q}, \mathbf{P})\,)$ that the new Hamiltonian H^* is constant in time, because this will entail simplification of the canonical equations for $\mathbf{Q}$ and $\mathbf{P}$. Especially convenient is to find a transformation that nullifies the new Hamiltonian $\mathcal{H}^*$, for in this case the new canonical equations will render the variables $(\,\mathbf{Q},\, \mathbf{P}\,)$ constant. One way of seeking such transformations is to consider S as a function of only $\mathbf{q}$, $\mathbf{P}$, and t . Under this assertion, the above equation will result in

$$\frac{\partial S}{\partial t}\, dt \;+\; \frac{\partial S}{\partial \mathbf{q}}\, d\mathbf{q} \;+\; \frac{\partial S}{\partial \mathbf{P}}\, d\mathbf{P} \;=\; \mathbf{p}^T\, d\mathbf{q} \;+\; \mathbf{Q}^T\, d\mathbf{P} \;-\; \left(\mathcal{H} \;-\; \mathcal{H}^*\right)\, dt \tag{47}$$

whence

$$\mathbf{p} \;=\; \frac{\partial S}{\partial \mathbf{q}} \quad , \qquad \mathbf{Q} \;=\; \frac{\partial S}{\partial \mathbf{P}} \quad , \qquad \mathcal{H} \;+\; \frac{\partial S}{\partial t} \;=\; \mathcal{H}^* \quad . \tag{48}$$

The function $S(\mathbf{q}\,,\; \mathbf{P}\,,\; t)$ can be then found by solving the Jacobi equation

$$\mathcal{H}\left(\mathbf{q},\, \frac{\partial S}{\partial \mathbf{q}}\,,\, t\right) \;+\; \frac{\partial S}{\partial t} \;=\; \mathcal{H}^*\left(\frac{\partial S}{\partial \mathbf{P}}\,,\, P\,,\, t\right) \quad . \tag{49}$$

In the free-spin case, the Jacobi equation becomes

$$\frac{1}{2\sin^2\theta}\left(\frac{\sin^2\psi}{I_1} + \frac{\cos^2\psi}{I_2}\right)\left(\frac{\partial \mathcal{S}}{\partial \phi} - \frac{\partial \mathcal{S}}{\partial \psi}\cos\theta\right)^2 + \frac{1}{2}\left(\frac{\cos^2\psi}{I_1} + \frac{\sin^2\psi}{I_2}\right)\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 \;+$$

$$\frac{1}{2I_3}\left(\frac{\partial \mathcal{S}}{\partial \psi}\right)^2 + \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\frac{\partial \mathcal{S}}{\partial \theta}\left(\frac{\partial \mathcal{S}}{\partial \phi} - \frac{\partial \mathcal{S}}{\partial \psi}\cos\theta\right)\frac{\sin\psi\,\cos\psi}{\sin\theta} + \frac{\partial \mathcal{S}}{\partial t} = \mathcal{H}^*\left(\frac{\partial S}{\partial \mathbf{P}}\,,\, \mathbf{P}\,,\, t\right) \quad . \tag{50}$$

At this point, Serret [19], Radau [20], and Tisserand [21] chose to put $\mathcal{H}^*$ equal to zero, thereby predetermining the new variables $(\,\mathbf{Q}\,,\; \mathbf{P}\,)$ to come out constants. (See, for example, equation (21) on page 382 in [21].) Thence the Jacobi equation (49) became equivalent to[4]

$$\mathcal{H}\left(\mathbf{q},\,\frac{\partial S}{\partial \mathbf{q}}\right)\;+\;\frac{dS}{dt}\;-\;\frac{\partial S}{\partial q}\,\frac{dq}{dt}\;=\;0\quad. \tag{51}$$

By taking into account that the initial Hamiltonian $\mathcal{H}$ depends explicitly neither on the time nor on the angle ϕ, Serret and his successors granted themselves an opportunity to seek the generating function in the simplified form of

$$S\;=\;A_1\,t\;+\;A_2\,\phi\;+\;\int\frac{\partial S}{\partial\theta}\,d\theta\;+\;\int\frac{\partial S}{\partial\psi}\,d\psi\;+\;C\quad, \tag{52}$$

the constant A_1 being equal to the negative value of the Hamiltonian $\mathcal{H}$, i.e., to the negative rotational kinetic energy:

$$A_1\;=\;-\,T_{kin}\quad. \tag{53}$$

The second constant, A_2, as well as the derivatives $\partial S/\partial\theta$ and $\partial S/\partial\psi$ can be calculated via the first formula (48):

$$A_2\;\equiv\;\frac{\partial S}{\partial\phi}\;=\;\Phi\quad,\quad\frac{\partial S}{\partial\theta}\;=\;\Theta\quad,\quad\frac{\partial S}{\partial\psi}\;=\;\Psi\quad, \tag{54}$$

so we get:

$$S\;=\;-\,t\,T_{kin}\;+\;\Phi\,\phi\;+\;\int\Theta\,d\theta\;+\;\int\Psi\,d\psi\;+\;C\quad, \tag{55}$$

Φ, Θ, and Ψ being given by formulae (40 - 42), and Φ being a constant of motion because $\mathcal{H}$ is ϕ-independent. Plugging of (40 - 42) into (55) yields:

$$S\;=\;-\,t\,T_{kin}\;+\;\Phi\,\phi\;+\;G\int\cos J\,d\psi\;+\;G\int\sin J\;\sin\left(\,l\;-\;\psi\,\right)\,d\theta\;+\;C$$

[4] The generating function should be written down exactly as (51) and **not** as

$$\mathcal{H}\left(\mathbf{q},\,\frac{\partial S}{\partial \mathbf{q}}\right)\;+\;\frac{dS}{dt}\;-\;\frac{\partial S}{\partial q}\,\frac{dq}{dt}\;-\;\frac{\partial S}{\partial P}\,\frac{dP}{dt}\;=\;0\quad,$$

because the new momenta P are not playing the role of independent variables but will emerge as constants in the solution.

$$= \, - \, t \, T_{kin} \, + \, G \, \phi \, \cos I \, + \, G \int \cos J \, \, dl \, +$$

$$G \int [- \, \cos J \, \, d \, (\, l \, - \, \psi \,) \, + \, \sin J \, \, \sin \, (\, l \, - \, \psi \,) \, d\theta \,] \, + \, C \quad , \tag{56}$$

The latter expression can be simplified through the equality (derived in the Appendix A.1)

$$d \, g \, - \, d \, (\phi - h) \, \, \cos I \, = \, - \, \cos J \, \, d \, (l \, - \, \psi) \, + \, \sin J \, \, \sin \, (l - \psi) \, d\theta \tag{57}$$

the angles $(\phi - h)$ and g being shown on Figs. 2, **??**. This will entail:

$$S \, = \, - \, t \, T_{kin} \, + \, G \, h \, \cos I \, + \, G \, g \, + \, G \int \cos J \, \, dl \, + \, C \quad . \tag{58}$$

Consider a coordinate system defined by the angular-momentum vector and a plane perpendicular thereto. This plane (in astronomy, called *invariable*) will be chosen as in Fig. 2 and Fig. **??**, so that the centre of mass of the body lie in this plane. The Euler angles ϕ_o , J , l defining the attitude of a body relative to the invariable plane obey the following differential relation:[5]

$$d\phi_o \, + \, \cos J \, dl \, = \, \frac{2 \, T_{kin}}{G} \, dt \quad . \tag{59}$$

Its integration results in the equality

$$\int_{\pi/2}^{l} \cos J \, dl \, = \, \frac{2 \, T_{kin}}{G} \, \left(\, t \, - \, t_o \, \right) \, - \, \phi_o \quad , \tag{60}$$

which is equivalent to

$$\int_{\pi/2}^{l} \cos J \, dl \, = \, \frac{2 \, T_{kin}}{G} \, \frac{u}{n} \, - \, \phi_o \quad , \tag{61}$$

[5] A somewhat tedious but elementary derivation of (59) is based on formulae connecting the angles ϕ_o , J , l with the components of the body angular velocity $\boldsymbol{\omega}$ and with the absolute value of the angular momentum (see [21], p.386):

$$I_1 \, \omega_1 \, = \, G \, \sin J \, \sin l \quad , \quad I_2 \, \omega_2 \, = \, G \, \sin J \, \cos l \quad , \quad I_3 \, \omega_3 \, = \, G \, \cos J \quad .$$

The Andoyer variables, which we shall introduce below, make (59) self-evident: (59) follows from equations (84), (97), (99) and from the observation that, for a free spin, $\dot{\phi}_o \, = \, \dot{g}$. One can also derive (59) by looking at the time rate of change of the projection one of the body's prinicpal momentum axes onto the invariable plane.

$n \;\equiv\; \sqrt{\left(\,G^2\,I_1^{-1}\;-\;2\,T_{kin}\,\right)\,\left(\,I_2^{-1}\;-\;I_3^{-1}\,\right)}$ being the mean angular velocity (or, playing an astronomical metaphor, the "mean motion") and u being the dimensionless time $u \;=\; n\;(t\;-\;t_o)$. At each instance of time, the value of u is unambiguously determined by the instantaneous attitude of the rotator, via the solution of the equations of motion (this solution is expressed through the elliptic functions of u). More specifically, u is a function of T_{kin} , G , $G\cos I$, θ, and ψ .

Insertion of (61) in (58) yields

$$S \;=\; \left(2\,\frac{u}{n}\;-\;t\right)\,T_{kin}\;+\;Gh\;\cos I\;\;+\;G\;\left(\,g\;-\;\phi_o\,\right)\,+\;C \tag{62}$$

It is now very tempting to state that, if we choose $\left(\,P_1\,,\;P_2\,,\;P_3\,\right)\;\equiv\;\left(\,G\,,\;G\;\cos I\;,\;T_{kin}\,\right)$, then expression (62) entails

$$\frac{\partial S}{\partial G}\;=\;Q_1 \tag{63}$$

and

$$\frac{\partial S}{\partial\,(G\;\cos I)}\;=\;Q_2 \tag{64}$$

where the two differences,

$$Q_1\;\equiv\;g\;-\;\phi_o \tag{65}$$

and

$$Q_2\;\equiv\;h \tag{66}$$

are integrals of motion, as can be easily seen in Fig.1. These two formulae, if correct, would implement our plan set out via the second equation (48).

In fact, equations (62 - 63) do **not** immediately follow from (61), because the variations of G , $G\;\cos I$, T_{kin} are **not** independent from the variations of the angles involved. These variations are subject to some constraints. A careful calculation, presented in the Appendix below, should begin with writing down an expression for δS , with the said constraints taken into account. This will give:

$$\delta S\;=\;\left(\frac{u}{n}\;-\;t\right)\;\delta T_{kin}\;+\;h\;\delta(\,G\;\cos I\,)\;+\;\left(\,g\;-\;\phi_o\,\right)\;\delta G \tag{67}$$

whence we deduce that (62 - 63) indeed are correct. We also see that the negative of the initial instant of time (which, too, is a trivial integral of motion), turns out to play the role of the third new coordinate:

$$\frac{\partial S}{\partial T_{kin}} \;=\; Q_3 \tag{68}$$

where

$$Q_3 \;\equiv\; \frac{u}{n} \;-\; t \;=\; -\,t_o \quad . \tag{69}$$

All in all, we have carried out a canonical transformation from the Euler angles and their conjugate momenta to the Serret variables $Q_1 \;=\; g \;-\; \phi_o$, $Q_2 \;=\; h$, $Q_3 \;=\; -\;t_o$ and their conjugate momenta $P_1 \;=\; G$, $P_2 \;=\; G\,\cos I$, $P_3 \;=\; T_{kin}$. In the modern literature, the constant P_2 is denoted with H . Constants h , G , H enter the Andoyer set of variables, discussed in the next subsection.

3.2 Andoyer (1923), and Deprit & Elipe (1993)

While Serret [19] had come up with a full reduction of the problem, Andoyer [17] suggested, through some geometric construction, a partial reduction. To understand the essence of that partial reduction, we shall start with a transformation from the inertial to the body frame via a coordinate system associated with the invariable plane.

Referring to Fig. 2, let OQ_1 and OQ_3 denote the LON's obtained from the intersection of the invariable plane with the inertial plane and with a plane fixed within the body, respectively. Let $\hat{\mathbf{i}}$ be a unit vector along the direction of OQ_1, and $\hat{\mathbf{j}}$ be a unit vector along OQ_3. Define a $3-1-3-1-3$ rotation sequence as follows:

- $R(h, \hat{\mathbf{s}}_3)$, a rotation about $\hat{\mathbf{s}}_3$ by $0 \leq h < 2\pi$, mapping $\hat{\mathbf{s}}_1$ onto $\hat{\mathbf{i}}$.
- $R(I, \hat{\mathbf{i}})$, a rotation about $\hat{\mathbf{i}}$ by $0 \leq I < \pi$, mapping $\hat{\mathbf{s}}_3$ onto the angular momentum vector, $\mathbf{G}$.
- $R(g, \mathbf{G}/G)$, a rotation about a unit vector pointing in the direction of the angular momentum by $0 \leq g < 2\pi$, mapping $\hat{\mathbf{i}}$ onto $\hat{\mathbf{j}}$.
- $R(J, \hat{\mathbf{j}})$, a rotation about $\hat{\mathbf{j}}$ by $0 < J < \pi$, mapping $\mathbf{G}$ onto $\hat{\mathbf{b}}_3$.

- $R(l, \hat{\mathbf{b}}_3)$, a rotation about $\hat{\mathbf{b}}_3$ by $0 \leq l \leq 2\pi$, mapping $\hat{\mathbf{j}}$ onto $\hat{\mathbf{b}}_1$.

The composite rotation may be written as

$$R(h, I, g, J, l) = R(l, \hat{\mathbf{b}}_3) R(J, \hat{\mathbf{j}}) R(g, \mathbf{G}/G) R(I, \hat{\mathbf{i}}) R(h, \hat{\mathbf{s}}_3) \tag{70}$$

Evaluation of the product of these five matrices will result in

$$R(h, I, g, J, l) = [\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3] \tag{71}$$

where

$$\mathbf{v}_1 = \begin{bmatrix} (\mathrm{c}_l \mathrm{c}_g - \mathrm{s}_l \mathrm{c}_J \mathrm{s}_g)\mathrm{c}_h - (\mathrm{c}_l \mathrm{s}_g + \mathrm{s}_l \mathrm{c}_J \mathrm{c}_g)\mathrm{c}_I - \mathrm{s}_l \mathrm{s}_J \mathrm{s}_I \mathrm{s}_h \\ -\mathrm{s}_l \mathrm{c}_g - \mathrm{c}_l \mathrm{c}_J \mathrm{s}_g \mathrm{c}_h - (-\mathrm{s}_l \mathrm{s}_g + \mathrm{c}_l \mathrm{c}_J \mathrm{c}_g)\mathrm{c}_I - \mathrm{c}_l \mathrm{s}_J \mathrm{s}_I \mathrm{s}_h \\ \mathrm{s}_J \mathrm{s}_g \mathrm{c}_h + (\mathrm{s}_J \mathrm{c}_g \mathrm{c}_I + \mathrm{c}_J \mathrm{s}_I)\mathrm{s}_h \end{bmatrix} \tag{72}$$

$$\mathbf{v}_2 = \begin{bmatrix} (\mathrm{c}_l \mathrm{c}_g - \mathrm{s}_l \mathrm{c}_J \mathrm{s}_g)\mathrm{s}_h + (\mathrm{c}_l \mathrm{s}_g + \mathrm{s}_l \mathrm{c}_J \mathrm{c}_g)\mathrm{c}_I - \mathrm{s}_l \mathrm{s}_J \mathrm{s}_I \mathrm{c}_h \\ -\mathrm{s}_l \mathrm{c}_g - \mathrm{c}_l \mathrm{c}_J \mathrm{s}_g \mathrm{s}_h + (-\mathrm{s}_l \mathrm{s}_g + \mathrm{c}_l \mathrm{c}_J \mathrm{c}_g)\mathrm{c}_I - \mathrm{c}_l \mathrm{s}_J \mathrm{s}_I \mathrm{c}_h \\ \mathrm{s}_J \mathrm{s}_g \mathrm{s}_h + (-\mathrm{s}_J \mathrm{c}_g \mathrm{c}_I - \mathrm{c}_J \mathrm{s}_I)\mathrm{c}_h \end{bmatrix} \tag{73}$$

$$\mathbf{v}_3 = \begin{bmatrix} (\mathrm{c}_l \mathrm{s}_g + \mathrm{s}_l \mathrm{c}_J \mathrm{c}_g)\mathrm{s}_I + \mathrm{s}_l \mathrm{s}_J \mathrm{c}_I \\ -\mathrm{s}_l \mathrm{s}_g + \mathrm{c}_l \mathrm{c}_J \mathrm{c}_g \mathrm{s}_I + \mathrm{c}_l \mathrm{s}_J \mathrm{c}_I \\ -\mathrm{s}_J \mathrm{c}_g \mathrm{s}_I + \mathrm{c}_J \mathrm{c}_I \end{bmatrix} \tag{74}$$

A sufficient condition for the transformation (43) to be canonical can be formulated in terms of perfect differentials. Given that the Hamiltonian lacks explicit time dependence, this condition will read[6] [26]:

$$\Phi d\phi + \Theta d\theta + \Psi d\psi = L dl + G dg + H dh \tag{75}$$

Let us first evaluate the left-hand side of (75). The differential of $R(\phi, \theta, \psi)$ is readily found to be [31]

$$dR(\phi, \theta, \psi) = \hat{\mathbf{s}}_3 d\phi + \hat{\mathbf{l}} d\theta + \hat{\mathbf{b}}_3 d\psi \tag{76}$$

Multiplying both sides of (76) by $\mathbf{G}$, and taking advantage of the identities (cf. (30 - 32)), one will arrive at

$$\mathbf{G} \cdot \hat{\mathbf{s}}_3 = \Phi \ , \quad \mathbf{G} \cdot \hat{\mathbf{l}} = \mathbf{G} \cdot (\hat{\mathbf{s}}_1 \mathrm{c}_\phi + \hat{\mathbf{s}}_2 \mathrm{s}_\phi) = \Theta \ , \quad \mathbf{G} \cdot \hat{\mathbf{b}}_3 = \Psi \ , \tag{77}$$

[6] We shall show shortly that indeed the perfect differentials criterion for canonical transformations holds in this case with the perfect differential of the generting function being identically equal to zero.

whence

$$\Phi\ d\phi\ +\ \Theta\ d\theta\ +\ \Psi d\psi\ =$$

$$\mathbf{G}\ \cdot\ \hat{\mathbf{s}}_3\ d\phi\ +\ \mathbf{G}\ \cdot\ (\hat{\mathbf{s}}_1\,\mathrm{c}_\phi\ +\ \hat{\mathbf{s}}_2\,\mathrm{s}_\phi)\ d\theta\ +\ \mathbf{G}\ \cdot\ \hat{\mathbf{b}}_3\ d\psi\ =\ \mathbf{G}\ \cdot\ dR\quad . \tag{78}$$

We shall now repeat the above procedure for the right-hand side of (75). The differential of $R(h, I, g, J, l)$ is evaluated similarly to (76) so as to get

$$dR(h\,,\ I\,,\ g\,,\ J\,,\ l)\ =\ \hat{\mathbf{s}}_3\ dh\ +\ \hat{\mathbf{i}}\ dI\ +\ \frac{\mathbf{G}}{G}\ dg\ +\ \hat{\mathbf{j}}\ dJ\ +\ \hat{\mathbf{b}}_3\ dl \tag{79}$$

Since

$$\mathbf{G}\cdot\hat{\mathbf{s}}_3 = \Phi, \quad \mathbf{G}\cdot\hat{\mathbf{i}} = 0, \quad \mathbf{G}\cdot\mathbf{G}/G = G, \quad \mathbf{G}\cdot\hat{\mathbf{j}} = 0, \quad \mathbf{G}\cdot\hat{\mathbf{b}}_3 = \Psi \quad , \tag{80}$$

then multiplication of both sides of (79) by $\mathbf{G}$ will lead us to

$$\mathbf{G}\cdot dR = \Phi dh + G dg + \Psi dl \tag{81}$$

Together, (78) and (81) give

$$\Phi\ d\phi\ +\ \Theta\ d\theta\ +\ \Psi\ d\psi\ =\ \mathbf{G}\ \cdot\ dR\ =\ \Phi\ dh\ +\ G\ dg\ +\ \Psi\ dl\quad , \tag{82}$$

comparison whereof with (75) immediately yields

$$\Phi = H \quad , \quad \Psi = L \quad . \tag{83}$$

To conclude, the transition from the Euler coordinates $\phi\,,\ \theta\,,\ \psi$ to the coordinates $h\,,\ g\,,\ l$ becomes a canonical transformation (with a vanishing generating function, as promised above), if we choose the momenta $H\,,\ G\,,\ L$ conjugated to $h\,,\ g\,,\ l$ as $H\,=\,\Phi\,,\ G\,=\,|\mathbf{G}|\,,\ L\,=\,\Psi$, correspondingly.

To get direct relations between the Euler angles and Andoyer angles $h\,,\ g\,,\ l$, we can compare the entries of $R(\phi, \theta, \psi)$ and $R(h, I, g, J, l)$. Alternatively, we may utilise the spherical laws of sines and cosines written for the spherical triangle $Q_1Q_2Q_3$ [8]. Adopting the former approach, we first express I and J in terms of the SA canonical momenta (cf. Fig. 2),

$$\mathrm{c}_I = H/G, \quad \mathrm{c}_J = L/G, \tag{84}$$

Equating the $(3,3)$ entries, the quotient of the $(1,3)$ and $(2,3)$ entries, and the quotient of the $(3,1)$ and the $(3,2)$ entries yields, respectively,

$$\mathrm{c}_\theta = \frac{LH}{G^2} - \frac{\mathrm{c}_g}{G^2}\sqrt{(G^2-L^2)(G^2-H^2)} \tag{85}$$

$$\tan\psi = \frac{\sqrt{G^2-H^2}(G\mathrm{s}_g\mathrm{c}_l + L\mathrm{s}_l\mathrm{c}_g) + H\mathrm{s}_l\sqrt{G^2-L^2}}{-\sqrt{G^2-H^2}(G\mathrm{s}_g\mathrm{s}_l + L\mathrm{c}_l\mathrm{c}_g) + H\mathrm{c}_l\sqrt{G^2-L^2}} \tag{86}$$

$$\tan\phi = -\frac{\sqrt{G^2-L^2}(G\mathrm{s}_g\mathrm{c}_h + H\mathrm{s}_h\mathrm{c}_g) + L\mathrm{s}_h\sqrt{G^2-H^2}}{\sqrt{G^2-L^2}(G\mathrm{s}_g\mathrm{s}_h - H\mathrm{c}_h\mathrm{c}_g) - L\mathrm{c}_h\sqrt{G^2-H^2}} \tag{87}$$

To complete the equations of transformation relating the Eulerian variables to the Andoyer variables, we utilise either Eqs. (90)-(92) or Eqs. (30)-(32) to calculate the magnitude of $\mathbf{G}$ and then express the momentum Θ in terms of the Andoyer variables. Carrying out this procedure yields

$$\Theta = G\mathrm{s}_J\mathrm{s}_{\psi-l} = \sqrt{G^2-L^2}\mathrm{s}_{\psi-l} \tag{88}$$

where the expressions for c_ψ and s_ψ may be readily obtained from Eq. (86).

Substituting the identities (83) and (85)-(88) into the (50) yields the new, single degree-of-freedom Hamiltonian

$$\mathcal{H}(g,h,l,G,H,L) = \frac{1}{2}\left(\frac{\mathrm{s}_l^2}{I_1} + \frac{\mathrm{c}_l^2}{I_2}\right)(G^2-L^2) + \frac{L^2}{2I_3} \tag{89}$$

Alternatively, we could have utilised the relations [18]

$$\begin{aligned} g_1 &= I_1\omega_1 = G\mathrm{s}_J\mathrm{s}_l = \sqrt{G^2-L^2}\mathrm{s}_l && (90)\\ g_2 &= I_2\omega_2 = G\mathrm{s}_J\mathrm{c}_l = \sqrt{G^2-L^2}\mathrm{c}_l && (91)\\ g_3 &= I_3\omega_3 = L && (92) \end{aligned}$$

to obtain the same result.

We would note that in the new Hamiltonian the coordinates g, h are cyclic, and hence the momenta G, H are integrals of motion. Also, since

$$-G \le L \le G, \tag{93}$$

the hamiltonian is non-negative,

$$\mathcal{H} \geq 0 \tag{94}$$

To get the canonical equations of motion, denote the generalised coordinates by $\boldsymbol{q} = [g, h, l]^T$ and the conjugate momenta by $\boldsymbol{p} = [G, H, L]^T$. Hamilton's equations in the absence of external torques are

$$\dot{\vec{q}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} \tag{95}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{q}} \tag{96}$$

Evaluating Hamilton's equations for (89), yields the canonical equations of free rotational motion:

$$\dot{g} = \frac{\partial \mathcal{H}}{\partial G} = G\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right) \tag{97}$$

$$\dot{h} = \frac{\partial \mathcal{H}}{\partial H} = 0 \tag{98}$$

$$\dot{l} = \frac{\partial \mathcal{H}}{\partial L} = L\left(\frac{1}{I_3} - \frac{\sin^2 l}{I_1} - \frac{\cos^2 l}{I_2}\right) \tag{99}$$

$$\dot{G} = -\frac{\partial \mathcal{H}}{\partial g} = 0 \tag{100}$$

$$\dot{H} = -\frac{\partial \mathcal{H}}{\partial h} = 0 \tag{101}$$

$$\dot{L} = -\frac{\partial \mathcal{H}}{\partial l} = (L^2 - G^2)\sin l\ \cos l\left(\frac{1}{I_1} - \frac{1}{I_2}\right) \tag{102}$$

Eqs. (99), (102) are separable differential equations, and hence can be solved in a *closed form* utilising the fact that $\mathcal{H}$ is constant (implying conservation of energy) [2]. The (l, L) phase plane may be characterised by plotting isoenergetic curves of the Hamiltonian (89). An example plot is depicted by Fig. 3, clearly showing the separatrix between rotational and librational motions.

4 THE CANONICAL PERTURBATION THEORY IN APPLICATION TO ATTITUDE DYNAMICS AND TO ROTATION OF CELESTIAL BODIES

The content of this section is based mainly on the results of paper by Efroimsky [32], to which we refer the reader for more details.

4.1 A modified Andoyer set of variables

To understand how the SA formalism may be used to model disturbing torques, let us start with an orbital dynamics analogy. In the theory of orbits, a Keplerian ellipse or hyperbola, emerging as an unperturbed two-body orbit, is considered as a sort of "elementary motion," so that all the other available trajectories are considered to be distortions of such conics, distortions implemented via endowing the orbital parameters C_j with their own time dependence. Points of the trajectory can be donated by the "elementary curves" either in a non-osculating manner, as in Fig. 4, or in the osculating one, as in Fig. 5.

Similarly, in attitude dynamics, a complex spin can be presented as a sequence of configurations constituted by some "elementary rotations." The easiest possibility is to use in this role the Eulerian cones, i.e., the loci of the spin axis, corresponding to undisturbed spin states. These are the simple motions exhibited by an undeformable unsupported rotator with no torques acting on it.[7] Then, to implement a perturbed mode, we shall have to go from one Eulerian cone to another, just as in Figs. 4 and 5 we go from one Keplerian conic to another. Accordingly, a smooth "walk" over the instantaneous Eulerian cones may be osculating or non-osculating.

The physical torques, the triaxiality of the rotator, and the fictitious torques caused by the frame noninertiality are among the possible disturbances causing this "walk." The latter two types of disturbances depend not only on the orientation but also on the angular velocity of the body.

[7] Here one opportunity is to use, as "elementary" motions, the non-circular Eulerian cones described by the actual triaxial top, when this top is unforced. Another opportunity is to use for this purpose the circular Eulerian cones described by a dynamically symmetrical top (and to consider its triaxiality as another perturbation). The results of our further study will be independent from the choice between these two options.

In the theory of orbits, we express the Lagrangian of the reduced two-body problem in terms of the spherical coordinates $q_j \;=\; \{\, r\,,\; \varphi\,,\; \theta\,\}$, then calculate the momenta p_j and the Hamiltonian $\mathcal{H}(q,\, p)$, and apply the Hamilton-Jacobi method [33] in order to get the Delaunay variables

$$\{\, P_1\,,\; P_2\,,\; P_3\;;\; Q_1\,,\; Q_2\,,\; Q_3\,\} \equiv \{\, L\,,\; G\,,\; H\;;\; l\,,\; g\,,\; h\,\} \;=\; \tag{103}$$
$$\{\, \sqrt{\mu a}\;\;,\;\;\sqrt{\mu a\,(1\,-\,e^2)}\;\;,\;\;\sqrt{\mu a\,(1\,-\,e^2)}\;\cos i\;\;;\;\;-\,M_o\;\;,\;\;-\,\omega\;\;,\;\;-\,\Omega\,\}\;,$$

μ being the reduced mass.

Very similarly, in attitude mechanics we specify a rotation state of a body by the three Euler angles $q_j \;=\; \phi\,,\,\theta\,,\,\psi$ and their momenta $\Phi\,,\;\Theta\,,\;\Psi$. After that, we can perform a canonical transformation to the afore described Serret variables (which are, in the unperturbed case, merely constants of motion). A different choice of the generating function would lead one to a different set of arbitrary constants, one consisting of the Andoyer variables' initial values: $\{\, L_o\,,\, G\,,\, H\,,\, l_o\,,\, g_o\,,\, h\,\}$, where $L_o\,,\; l_o$ and g_o are the initial values of $L\,,\; l$ and g . The latter set[8] (which we shall call "modified Andoyer set") consists only of constants of integration, and hence the corresponding Hamiltonian becomes nil. Therefore, these constants are the true analogues of the Delaunay set with M_o (while the conventional Andoyer set is analogous to the Delaunay set with M used instead of M_o .). The main result obtained below for the modified Andoyer set $\{\, L_o\,,\, G\,,\, H\,,\, l_o\,,\, g_o\,,\, h\,\}$ will then be easily modified for the conventional Andoyer set of variables $\{\, L\,,\, G\,,\, H\,,\, l\,,\, g\,,\, h\,\}$.

All in all, the canonical treatment of both orbital and rotational cases begins with

$$\dot{\mathbf{q}} \;=\; \frac{\partial \mathcal{H}^{(o)}}{\partial \mathbf{p}}\quad,\qquad \dot{\mathbf{p}} \;=\; -\,\frac{\partial \mathcal{H}^{(o)}}{\partial \mathbf{q}}\quad, \tag{104}$$

$\mathbf{q}$ and $\mathbf{p}$ being the coordinates and their conjugated momenta, in the orbital case, or the Euler angles and their momenta, in the rotation case. Then one switches, by a canonical transformation

$$\mathbf{q} \;=\; f(\mathbf{Q}\,,\; \mathbf{P}\,,\; t) \tag{105}$$
$$\mathbf{p} \;=\; \chi(\mathbf{Q}\,,\; \mathbf{P}\,,\; t)\;,$$

[8] A similar set consisting of the initial values of Andoyer-type variables was pioneered by Fukushima and Ishizaki [34].

to

$$\dot{\mathbf{Q}} \;=\; \frac{\partial \mathcal{H}^*}{\partial \mathbf{P}} \;=0 \quad , \qquad \dot{\mathbf{P}} \;=\; -\,\frac{\partial \mathcal{H}^*}{\partial \mathbf{Q}} \;=\; 0 \quad , \qquad \mathcal{H}^* \;=\; 0 \;\;, \tag{106}$$

where $\mathbf{Q}$ and $\mathbf{P}$ are the Delaunay variables, in the orbital case, or the (modified, as explained above) Andoyer variables $\{L_o\,,\;G\,,\;H\,,\;l_o\,,\;g_o\,,\;h\}$, in the attitude case.

This algorithm is based on the circumstance that an unperturbed Kepler orbit (and, similarly, an undisturbed Euler cone) can be fully defined by six parameters so that:

1. These parameters are canonical variables $\{\mathbf{Q}\,,\,\mathbf{P}\}$ with a zero Hamiltonian: $\mathcal{H}^*(\mathbf{Q},\,\mathbf{P}) \;=\; 0$; and therefore these parameters are constants.

2. For constant $\mathbf{Q}$ and $\mathbf{P}$, the transformation equations (105) are equivalent to the equations of motion (104).

4.2 The canonical treatment of perturbations

Under perturbations, the "constants" $\mathbf{Q}\,,\,\mathbf{P}$ begin to evolve, so that after their substitution into

$$\begin{aligned} \mathbf{q} \;&=\; f\,(\,\mathbf{Q}(t)\,,\;\mathbf{P}(t)\,,\;t\,) \\ \mathbf{p} \;&=\; \chi(\,\mathbf{Q}(t)\,,\;\mathbf{P}(t)\,,\;t\,) \end{aligned} \tag{107}$$

(f and χ being the same functions as in (105)), the resulting motion obeys the disturbed equations

$$\dot{\mathbf{q}} \;=\; \frac{\partial\left(\mathcal{H}^{(o)}\,+\,\Delta\mathcal{H}\right)}{\partial \mathbf{p}} \quad , \qquad \dot{\mathbf{p}} \;=\; -\;\frac{\partial\left(\mathcal{H}^{(o)}\,+\,\Delta\mathcal{H}\right)}{\partial \mathbf{q}} \quad . \tag{108}$$

We want our "constants" $\mathbf{Q}$ and $\mathbf{P}$ also to remain canonical and to obey

$$\dot{\mathbf{Q}} \;=\; \frac{\partial\,(\mathcal{H}^*\,+\,\Delta\mathcal{H}^*)}{\partial \mathbf{P}} \quad , \qquad \dot{\mathbf{P}} \;=\; -\,\frac{\partial\,(\mathcal{H}^*\,+\,\Delta\mathcal{H}^*)}{\partial \mathbf{Q}} \tag{109}$$

where

$$\mathcal{H}^* \;=\; 0 \quad \text{and} \quad \Delta\mathcal{H}^*\,(\mathbf{Q}\,,\;\mathbf{P}\;\;t) \;=\; \Delta\mathcal{H}\,(\,\mathbf{q}(\mathbf{Q},\mathbf{P},t)\,,\;\mathbf{p}(\mathbf{Q},\mathbf{P},t)\,,\;t\,)\;. \tag{110}$$

Above all, it is often desired that the perturbed "constants" $C_j \equiv Q_1 , Q_2 , Q_3 , P_1 , P_2 , P_3$ (the Delaunay variables, in the orbital case, or the modified Andoyer variables, in the rotation case) remain osculating. This demand means that the perturbed velocity should be expressed by the same function of $C_j(t)$ and t as the unperturbed velocity used to. In other words, the instantaneous "simple motions" parameterised by these constants should be tangent to the perturbed trajectory. (In the orbital case, this situation is shown on Fig. 5.) Let us check if osculation is always preserved under perturbation. The perturbed velocity reads

$$\dot{\mathbf{q}} \;=\; \mathbf{g} \;+\; \mathbf{\Phi} \tag{111}$$

where

$$\mathbf{g}(C(t),\, t) \;\equiv\; \frac{\partial \mathbf{q}(C(t),\, t)}{\partial t} \tag{112}$$

is the functional expression for the unperturbed velocity; and

$$\mathbf{\Phi}(C(t),\, t) \;\equiv\; \sum_{j=1}^{6} \frac{\partial q(C(t),\, t)}{\partial C_j}\, \dot{C}_j(t) \tag{113}$$

is the convective term. Since we chose the "constants" C_j to make canonical pairs $(\mathbf{Q},\, \mathbf{P})$ obeying (109) - (110) with vanishing $\mathcal{H}^*$, then insertion of (109) into (113) will result in

$$\mathbf{\Phi} \;=\; \sum_{n=1}^{3} \frac{\partial \mathbf{q}}{\partial Q_n}\, \dot{\mathbf{Q}}_n(t) \;+\; \sum_{n=1}^{3} \frac{\partial \mathbf{q}}{\partial P_n}\, \dot{\mathbf{P}}_n(t) \;=\; \frac{\partial \Delta \mathcal{H}(\mathbf{q},\, \mathbf{p})}{\partial \mathbf{p}} \quad . \tag{114}$$

So the canonicity demand is often incompatible with osculation. Specifically, whenever a momentum-dependent perturbation is present, we still can use the ansatz (107) for calculation of the coordinates and momenta, but can no longer use (112) for calculating the velocities. Instead, we must use (111). Application of this machinery to the case of orbital motion is depicted on Fig.4. Here the constants $C_j \;=\; (Q_n,\, P_n)$ parameterise instantaneous ellipses that, for nonzero $\mathbf{\Phi}$, are *not* tangent to the trajectory. (For more details see Efroimsky & Goldreich [35], [36], and Efroimsky [37].) In the case of rotational motion, the situation will be identical, except that, instead of instantaneous Keplerian conics, one will deal with instantaneous Eulerian cones (i.e., with the loci of the rotational axis, corresponding to non-perturbed spin states), as elaborated below.

4.3 From the modified Andoyer variables to the regular ones

Practical calculations used in the theory of planetary rotation and in spacecraft attitude dynamics are almost always set out in terms of the regular Andoyer variables, not in terms of their initial values (the paper by Fukushima & Ishizaki [34] being a unique exception). Fortunately, all our gadgetry, developed above for the modified Andoyer set, stays applicable for the regular set. To prove this, let us consider the unperturbed parameterisation of the Euler angles $q_n \;=\; (\,\phi\,,\,\theta\,,\,\psi\,)$ via the regular Andoyer variables $A_j \;=\; (\,l\,,\;g\,,\;h\,;\;L\,,\,G\,,\,H\,)$:

$$q_n \;=\; f_n\,(\;A_1(C\,,\,t)\;,\;\ldots\;,\;A_6(C\,,\,t)\;)\quad, \tag{115}$$

each element A_i being a function of time and of the initial values $C_j = (l_o\,,\,g_o\,,\,h\,;\,L_o\,,\,G\,,\,H\,)$. The explicit form of parameterisation (115) is given by (85 - 87). When a perturbation gets turned on, this parameterisation stays, while the time evolution of the elements A_i changes: beside the standard time-dependence inherent in the free-spin Andoyer variables, the perturbed elements acquire an extra time-dependence through the evolution of their initial values.[9] Then the time evolution of an Euler angle $q_n = (\,\phi\,,\;\theta\,,\;\psi\,)$ will be given by a sum of two items: (1) the angle's unperturbed dependence upon time and time-dependent Andoyer variables; and (2) the convective term Φ_n that arises from a perturbation-caused alteration of the Andoyer variables' dependence upon the time:

$$\dot{q}_n \;=\; \mathrm{g}_n \;+\; \Phi_n\quad. \tag{116}$$

The unperturbed part is

$$\mathrm{g}_n = \sum_{i=1}^{6}\frac{\partial f_n}{\partial A_i}\,\left(\frac{\partial A_i}{\partial t}\right)_C\quad, \tag{117}$$

[9] This is fully analogous to the transition from the unperturbed mean longitude,

$$M(t) \;=\; M_o \;+\; n\;(\,t\;-\;t_o\,)\quad,\quad\text{with}\quad M_o\,,\;n\,,\;t_o\;=\;const\quad,$$

to the perturbed one,

$$M(t) \;=\; M_o(t) \;+\; \int_{t_o}^{t} n(t')\;dt'\quad,\quad\text{with}\quad t_o\;=\;const\quad,$$

in orbital dynamics.

while the convective term is given by

$$\Phi_n = \sum_{i=1}^{6}\sum_{j=1}^{6}\left(\frac{\partial f_n}{\partial A_i}\right)_t\left(\frac{\partial A_i}{\partial C_j}\right)_t \dot{C}_j = \sum_{j=1}^{6}\left(\frac{\partial f_n}{\partial C_j}\right)_t \dot{C}_j$$

$$= \sum_{j=1}^{3}\left(\frac{\partial f_n}{\partial Q_j}\right)_t \dot{Q}_j + \sum_{j=1}^{3}\left(\frac{\partial f_n}{\partial P_j}\right)_t \dot{P}_j = \frac{\partial \Delta\mathcal{H}(q,p)}{\partial p_n} \quad , \qquad (118)$$

where the set C_j is split into canonical coordinates and momenta like this: $Q_j = (l_o, g_o, h)$ and $P_j = (L_o, G, H)$. In the case of free spin they obey the Hamilton equations with a vanishing Hamiltonian and, therefore, are all constants. In the case of disturbed spin, their evolution is governed by (109 - 110), substitution whereof in (118) once again takes us to (114). This means that the nonosculation-caused convective corrections to the velocities stay the same, no matter whether we parameterise the Euler angles through the modified Andoyer variables (variable constants) or through the regular Andoyer variables. This invariance will become obvious if we consider the analogy with orbital mechanics: in Fig. 4, the correction $\mathbf{\Phi}$ is independent of how we choose to parameterise the non-osculating instantaneous ellipse – through the Delaunay set with M_o or through the one containing M .

4.4 The Andoyer variables introduced in a precessing frame of reference

4.4.1 Physical motivation

Let us consider a case when the perturbing torque depends not only on the instantaneous orientation but also on the instantaneous angular velocity of the rotator. In particular, we shall be interested in the fictitious torque emerging when the description is carried out in a precessing coordinate system. This situation is often encountered in the theory of planetary rotation, where one has to describe a planet's spin not in inertial axes but relative to a frame associated with the planet's circumsolar orbital plane (the planet's ecliptic). The latter frame is noninertial, because the ecliptic is always precessing due to the perturbations exerted by the other planets. The reason why astronomers need to describe the planet's rotation not in an inertial frame but in the ecliptic one is that this description provides the history of the planet's obliquity, i.e., of the equator's inclination on the ecliptic. As the obliquity determines the latitudinal distribution of insolation, the long-term history of the obliquity is a key to understanding the climate evolution. Interestingly, the climate is much more sensitive to the obliquity of the planet than to the eccentricity of its orbit. (Murray et al [38], Ward [39], [40])

The canonical theory of rotation of a rigid body in a precessing coordinate frame was pioneered by Giacaglia and Jefferys [27]. It was based on the Andoyer variables and was used by the authors to describe rotation of a space station. This theory was greatly furthered by Kinoshita [8] who applied it to rotation of the rigid Earth. Later it was extended by Getino, Ferrandiz and Escapa [9], [10], [11] to the case of nonrigid Earth. Simplified versions of the Kinoshita theory were employed by Laskar and Robutel [14] and by Touma and Wisdom [15, 16] in their studies of the long-term evolution of the obliquity of Mars. While a detailed explanation of this line of research will require a separate review paper, here we shall very briefly describe the use of the Andoyer variables in the Kinoshita theory, and shall dwell, following Efroimsky [32], on the consequences of these variables being nonosculating.

4.4.2 Formalism

Consider an unsupported rigid body whose spin should be described with respect to a coordinate system, which itself is precessing relative to some inertial axes. The said system is assumed to precess at a rate $\boldsymbol{\mu}$ so the kinetic energy of rotation, in the inertial axes, is given by

$$T_{kin} \;=\; \frac{1}{2}\;\sum_{i=1}^{3}\;I_i\;\left(\,\omega_i\;+\;\mu_i\,\right)^2 \tag{119}$$

where $\boldsymbol{\omega}$ is the body-frame-related angular velocity of the body relative to the precessing coordinate system, while $\boldsymbol{\mu}$ is the angular velocity relative to the inertial frame. In (119), both $\boldsymbol{\omega}$ and $\boldsymbol{\mu}$ are resolved into their components along the principal axes. The role of canonical coordinates will be played the Euler angles $q_n \;=\; \phi\,,\;\theta\,,\;\psi$ that define the orientation of the principal body basis relative to the *precessing* coordinate basis. To compute their conjugate momenta $p_n \;=\; \Phi\,,\;\Theta\,,\;\Psi$, let us assume that noninertiality of the precessing coordinate system is the only angular-velocity-dependent perturbation. Then the momenta are simply the derivatives of the kinetic energy. With aid of (5 - 7), they can be written as

$$\Phi = \frac{\partial T_{kin}}{\partial \dot{\phi}} = I_1\,(\omega_1 + \mu_1)\sin\theta\,\sin\psi + I_2\,(\omega_2 + \mu_2)\sin\theta\,\cos\psi + I_3\,(\omega_3 + \mu_3)\cos\theta \quad , \qquad (120)$$

$$\Theta \;=\; \frac{\partial T_{kin}}{\partial \dot{\theta}} \;=\; I_3\;(\,\omega_3\,+\,\mu_3\,) \quad , \qquad (121)$$

$$\Psi \;=\; \frac{\partial T_{kin}}{\partial \dot{\psi}} \;=\; I_1\;(\,\omega_1\,+\,\mu_1)\;\cos\psi\;-\;I_2\;(\,\omega_2\,+\,\mu_2)\;\sin\psi \quad . \qquad (122)$$

These formulae enable one to express the angular-velocity components ω_i and the derivatives $\dot{q}_n \,=\, (\,\dot{\phi}\,,\,\dot{\theta}\,,\,\dot{\psi}\,)$ via the momenta $p_n \,=\, (\,\Phi\,,\,\Theta\,,\,\Psi\,)$. Insertion of (120 - 122) into

$$\mathcal{H} \,=\, \sum_n \,\dot{q}_n\,p_n \,-\, \mathcal{L} \,=\, \dot{\phi}\,\Phi \,+\, \dot{\theta}\,\Theta \,+\, \dot{\psi}\,\Psi \,-\, T \,+\, V(\phi\,,\;\theta\,,\;\psi) \qquad (123)$$

results, after some lengthy algebra, in

$$\mathcal{H} \,=\, T \,+\, \Delta\mathcal{H} \qquad (124)$$

where

$$\begin{aligned}\Delta\mathcal{H} \,=\, -\,\mu_1\,&\left[\frac{\sin\psi}{\sin\theta}\;(\Phi\,-\,\Psi\,\cos\theta)\,+\,\Theta\,\cos\psi\right]\\ -\,\mu_2\,&\left[\frac{\cos\psi}{\sin\theta}\;(\Phi\,-\,\Psi\,\cos\theta)\,-\,\Theta\,\sin\psi\right]\\ -\,\mu_3\,\Psi\,&+\,V(\phi\,,\;\theta\,,\;\psi) \quad , \qquad (125)\end{aligned}$$

and the potential V is presumed to depend only upon the angular coordinates, not upon the momenta.

Now let us employ the machinery described in the preceding subsection. The Euler angles connecting the body axes with the precessing frame will now be expressed via the Andoyer variables by means of (115). (The explicit form of the functional dependence (115) is given by (85 - 87), but this exact form is irrelevant to us.) The fact that the Andoyer variables are introduced in a noninertial frame is accounted for by the emergence of the μ-terms in the expression (125) for the disturbance $\Delta\mathcal{H}$. Insertion of (125) into (118) entails:

$$\dot{q}_n \;=\; \mathrm{g}_n \,+\, \frac{\partial \Delta\mathcal{H}}{\partial p_n} \tag{126}$$

the convective terms being given by

$$\frac{\partial \Delta\mathcal{H}}{\partial P_1} \;=\; \frac{\partial \Delta\mathcal{H}}{\partial \Phi} \;=\; -\;\frac{\mu_1\;\sin\psi \;+\; \mu_2\;\cos\psi}{\sin\theta} \quad , \tag{127}$$

$$\frac{\partial \Delta\mathcal{H}}{\partial P_2} \;=\; \frac{\partial \Delta\mathcal{H}}{\partial \Theta} \;=\; -\;\mu_1\;\cos\psi \;+\; \mu_2\;\sin\psi \quad , \tag{128}$$

$$\frac{\partial \Delta\mathcal{H}}{\partial P_3} \;=\; \frac{\partial \Delta\mathcal{H}}{\partial \Psi} \;=\; (\,\mu_1\;\sin\psi \;+\; \mu_2\;\cos\psi\,)\;\cot\theta \;-\; \mu_3 \quad . \tag{129}$$

It should be stressed once again that the indices $n \;=\; 1\,,\,2\,,\,3$ in (126) number the Euler angles, so that $\dot{q}_n$ stand for $\dot{\phi}\,,\,\dot{\theta}\,,\,\dot{\psi}$, and p_n signify $\Phi\,,\,\Theta\,,\,\Psi$. At the same time, the subscripts $i \;=\; 1\,,\,2\,,\,3$ accompanying the components of $\boldsymbol{\mu}$ in (127) correspond to the principal body axes.

4.4.3 The physical interpretation of the Andoyer variables defined in a precessing frame.

The physical content of the Andoyer construction built in inertial axes is transparent: by definition, the element G is the magnitude of the angular-momentum vector, L is the projection of the angular-momentum vector on the principal axis $\hat{\mathbf{b}}_3$ of the body, while H is the projection of the angular-momentum vector on the $\hat{\mathbf{s}}_3$ axis of the inertial coordinate system. The variable h conjugate to H is the angle from the inertial reference longitude to the ascending node of the invariable plane (the one perpendicular to the angular momentum). The variable g conjugate to G is the angle from the ascending node of the invariable plane on the reference plane to the ascending node of the equator on the invariable plane. Finally, the variable conjugate to L is the angle l from the ascending node of the equator on the invariable plane to the the $\hat{\mathbf{b}}_1$ body axis. Two auxiliary quantities defined through

$$\cos I \;=\; \frac{H}{G} \quad , \qquad \cos J \;=\; \frac{L}{G} \quad ,$$

too, have evident physical meaning: I is the angle between the angular-momentum vector and the $\hat{\mathbf{s}}_3$ space axis, while J is the angle between the angular-momentum vector and the $\hat{\mathbf{b}}_3$ principal axis of the body, as can be seen on Fig. 2.

Will all the Andoyer variables and the auxiliary angles I and J retain the same physical meaning if we re-introduce the Andoyer construction in a noninertial frame? The answer is affirmative, because a transition to a noninertial frame is no different from any other perturbation: precession of the fiducial frame $(\hat{\mathbf{s}}_1\,,\,\hat{\mathbf{s}}_2\,,\,\hat{\mathbf{s}}_3)$ is equivalent to emergence of an extra perturbing torque, one generated by the inertial forces. In the original Andoyer construction assembled in an inertial space, the invariable plane was orthogonal to the instantaneous direction of the angular-momentum vector: If the perturbing torques were to instantaneously vanish, the angular-momentum vector (and the invariable plane orthogonal thereto) would freeze in their positions relative to the fiducial axes $(\hat{\mathbf{s}}_1\,,\,\hat{\mathbf{s}}_2\,,\,\hat{\mathbf{s}}_3)$ (which were inertial and therefore indifferent to vanishing of the perturbation). Now, that the Andoyer construction is built in a precessing frame, the fiducial plane is no longer inertial. Nevertheless if the inertial torques were to instantaneously vanish, then the invariable plane would still freeze relative to the fiducial plane (because the fiducial plane would cease its precession). Therefore, all the will variables retain their initial meaning. In particular, the variables I and J defined as above will be the angles that the angular-momentum makes with the precessing $\hat{\mathbf{s}}_3$ space axis and with the $\hat{\mathbf{b}}_3$ principal axis of the body, correspondingly.

4.4.4 Calculation of the angular velocities via the Andoyer variables introduced in a precessing frame of reference

Let us now get back to formulae (5 - 7) for the principal-body-axes-related components of the angular velocity. These formulae give this angular velocity as a function of the rates of Euler angle's evolution, so one can symbolically denote the functional dependence (5 - 7) as $\boldsymbol{\omega}\,=\,\boldsymbol{\omega}(\dot{q})$. This dependence is linear, so

$$\boldsymbol{\omega}\,(\,\dot{q}(A)\,)\;=\;\boldsymbol{\omega}(\,\mathrm{g}(A)\,)\;+\;\boldsymbol{\omega}\,(\;\partial\Delta\mathcal{H}/\partial p\;)\quad, \tag{130}$$

A being the set of Andoyer variables. Direct substitution of (127 - 129) into (5 - 7) will then show that the second term on the right-hand side in (130) is exactly $-\boldsymbol{\mu}$:

$$\boldsymbol{\omega}(\dot{q}(A)) \;=\; \boldsymbol{\omega}(\mathrm{g}) \;-\; \boldsymbol{\mu} \quad . \tag{131}$$

Since the "total" angular velocity $\boldsymbol{\omega}(\dot{q})$ is that of the body frame relative to the precessing frame, and since $\boldsymbol{\mu}$ is that of the precessing frame relative to some inertial frame, then $\boldsymbol{\omega}(\,\mathrm{g}(A)\,)$ *will always return the angular velocity of the body relative to the inertial frame of reference, despite the fact that the Andoyer variables A were introduced in a precessing frame.* This nontrivial fact has immediate ramifications for the theory of planetary rotation. These will be considered in the subsequent subsections.

To better understand the origin of the said fact, let us again get back to the basic Andoyer formalism introduced in the previous subsection. As the first step, we introduce, in an unperturbed setting (i.e., for an unsupported rigid rotator considered in an inertial frame), the parameterisation of the Euler angles $q_n \;=\; (\,\phi\,,\,\theta\,,\,\psi\,)$ through the Andoyer variables $A_j \;=\; (\,l\,,\;g\,,\;h\,;\;L\,,\,G\,,\,H\,)$:

$$q_n \;=\; f_n\,(\,A_1(C\,,\,t)\,,\;\ldots\,,\;A_6(C\,,\,t)\,) \quad , \tag{132}$$

each variable A_i being dependent upon its initial value C_i and the time:

$$A_i \;=\; C_i \;+\; n_i\,(t\;-\;t_o) \tag{133}$$

where the mean motions[10] n_i may bear dependence upon the other Andoyer variables A_j . As already mentioned above, this is analogous to the evolution of the mean longitude $M \;=\; M_o \;+\; n\,(t\;-\;t_o)$ in the undisturbed two-body problem (where the mean motion n is a function of another orbital element, the semimajor axis a).

The initial values C_i are integration constants, so in the unperturbed case one can calculate the velocities $\dot{q}_n \;=\; \left(\dot{\phi}\,,\,\dot{\theta}\,,\,\dot{\psi}\right)$ simply as the partial derivatives

$$\mathrm{g}_n(A) \;\equiv\; \left(\frac{\partial q_n(A(C,\,t))}{\partial t}\right)_C \;\equiv\; \sum_i \frac{\partial q_n(A)}{\partial A_i}\left(\frac{\partial A_i(C,\,t)}{\partial t}\right)_C \tag{134}$$

As the second step, we employ this scheme in a perturbed setting. In particular, we introduce a perturbation caused by our transition to a coordinate system precessing at a rate $\boldsymbol{\mu}$. Our Euler angles now describe the

[10] Three of the six Andoyer variables have vanishing n_i in the unperturbed free-spin case, but at this point it is irrelevant.

body orientation relative to this precessing frame. By preserving the parameterisation (132), we now introduce the Andoyer variables A_i in this precessing frame. Naturally, the time evolution of A_i changes because the frame-precession-caused Hamiltonian disturbance $\Delta\mathcal{H}$ now shows itself in the equations of motion. This disturbance depends not only on the body's orientation but also on its angular velocity, and therefore our Andoyer variables cannot be osculating (see equation (114) and the paragraph thereafter). This means that the unperturbed velocities, i.e., the partial derivatives (134) no longer return the body's angular velocity relative to the precessing frame (i.e., relative to the frame wherein the Andoyer variables were introduced). This angular velocity is rather given by the sum (116). However, as explained above, the unperturbed expression $\mathrm{g}(A)$, too, has a certain physical meaning: when plugged into $\boldsymbol{\omega}(\mathrm{g})$, it always returns the angular velocity *in the inertial frame*. It does so even despite the fact that now the Andoyer parameterisation is introduced in a precessing coordinate frame.[11]

[11] This parallels a situation in orbital dynamics, where the role of canonical variables is played by the Delaunay constants $C\,=\,(\,Q\,;\;P\,)\,=\,(\,L\,,\,G\,,\,H\,;\;-\,M_o\,,\,-\,\omega\,,\,-\,\Omega\,)$ where

$$L\,\equiv\,\mu^{1/2}\,a^{1/2}\quad,\quad G\,\equiv\,\mu^{1/2}\,a^{1/2}\,\left(1\,-\,e^2\right)^{1/2}\quad,\quad H\,\equiv\,\mu^{1/2}\,a^{1/2}\,\left(1\,-\,e^2\right)^{1/2}\cos i\quad,$$

the parameters $a,\,e,\,i,\,\omega,\,\Omega,\,M_o$ being the Kepler orbital variables. In the unperturbed setting (the two-body problem in inertial axes), the Cartesian coordinates $\boldsymbol{r}\,\equiv\,(\,x_1\,,\,x_2\,,\,x_3)$ and velocities $(\,\dot{x}_1\,,\,\dot{x}_2\,,\,\dot{x}_3)$ are expressed via the time and the Delaunay constants by means of the following functional dependencies:

$$\boldsymbol{r}\,=\,\boldsymbol{f}\,(\,C\,,\;t\,)\qquad\text{and}\qquad\mathbf{v}\,=\,\mathbf{g}\,(\,C\,,\;t\,)\quad,\quad\text{where}\quad\mathbf{g}\,\equiv\,\partial\boldsymbol{f}/\partial t\quad.$$

If we want to describe a satellite orbiting a precessing oblate planet, we may fix our reference frame on the precessing equator of date. Then the two-body problem will get amended with two disturbances. One, $\Delta\mathcal{H}_{oblate}$, caused by the presence of the equatorial bulge of the planet, will depend only upon the satellite's position. Another one, $\Delta\mathcal{H}_{precess}$, will stem from the noninertial nature of our frame and, thus, will give birth to velocity-dependent inertial forces. Under these perturbations, the Delaunay constants will become canonical variables evolving in time. As explained in subsection 4.2, the velocity-dependence of one of the perturbations involved will make the Delaunay variables nonosculating. On the one hand, the expression $\boldsymbol{r}\,=\,\boldsymbol{f}\,(\,C(t)\,,\;t\,)$ will return the correct Cartesian coordinates of the satellite in the precessing equatorial frame, i.e., in the frame wherein the Delaunay variables were introduced. On the other hand, the expression $\mathbf{g}\,(\,C\,,\;t\,)$ will no longer return the correct velocities in that frame. Indeed, according to (111 - 114), the Cartesian components of the velocity in the precessing equatorial frame will be given by $\mathbf{g}\,(\,C\,,\;t\,)\,+\,\partial\Delta\mathcal{H}_{precess}/\partial p$. However, it turns out that $\mathbf{g}\,(\,C\,,\;t\,)$ renders the velocity with respect to the inertial frame of reference. (Efroimsky [37], [41])

4.4.5 Example 1. The theory of Earth rotation

As an example, let us consider the rigid-Earth-rotation theory by Kinoshita [8]. Kinoshita began with the standard Andoyer formalism in inertial axes. He explicitly wrote down the expressions (115) for the Euler angles $q_n = f_n(A)$ of the figure axis of the Earth, differentiated them to get the expressions for the velocities $\dot{q}_n$ as functions of the Andoyer variables: $\dot{q}_n = \mathrm{g}_n(A)$, and then used those expressions to write down the angles $I_r(\mathrm{g}(A))$ and $h_r(\mathrm{g}(A))$ that define the orientation of the rotation axis.[12] Kinoshita pointed out that one's knowledge of the solar and lunar torques, exerted on the Earth due to its nonsphericity, would enable one to write down the appropriate Hamiltonian perturbations and to calculate, by the Hori-Deprit method ([3] - [7]), the time evolution of the Andoyer variables. Substitution thereof into (132) would then give the time evolution of the Euler angles that define the figure axis of the planet. Similarly, substitution of the calculated time dependencies of the Andoyer variables A into the expressions for $I_r(\mathrm{g}(A))$ and $h_r(\mathrm{g}(A))$ would yield the time evolution of the planet's rotation axis.

The situation, however, was complicated by the fact that Kinoshita's goal was to calculate the dynamics relative to the precessing ecliptic plane. To achieve the goal, Kinoshita amended the afore described method by adding, to the lunisolar perturbations, the momentum-dependent frame-precession-caused term (see our formula (125) above). Stated differently, he introduced the Andoyer variables in a precessing frame of reference. This made his Andoyer variables A nonosculating. Kinoshita missed this circumstance and went on to calculate the time dependence of the so introduced Andoyer variables. Then he plugged those into the expressions $q_n = f_n(A)$ for the Euler angles of the figure and into the expressions $I_r(\mathrm{g}(A))$ and $h_r(\mathrm{g}(A))$ for the orientation angles of the Earth rotation axis. The expressions $q_n = f_n(A)$ still gave him the correct Euler angles of the Earth figure (now, relative to the precessing ecliptic plane). The expressions $I_r(\mathrm{g}(A))$ and $h_r(\mathrm{g}(A))$ did *NOT* give him the correct orientation of the angular-velocity vector relative to the precessing frame, because the Andoyer variables introduced by Kinoshita in the precessing frame were nonosculating. This potential drawback of his theory have long been ignored, because no direct methods of measurement of the angular velocity of the Earth had been developed until 2004.

[12] For the Euler angles, Kinoshita chose notations, which are often used by astronomers: h , I , ϕ and which are different from the convention $q_n = \phi\,,\,\theta\,,\,\psi$ used in physics.

While the thitherto available observations referred only to the orientation of the Earth figure (Kinoshita [42]), a technique based on ring laser gyroscopes provided a direct measurement of the instantaneous angular velocity of the Earth *relative to an inertial frame*. (Schreiber et al. [43])

As we demonstrated above, when the Andoyer variables A are introduced in a precessing frame, the expressions $q_n = f_n(A)$ return the Euler angles of the body *relative to this precessing frame*, while $\boldsymbol{\omega}(\mathrm{g}_n(A))$ returns the body-frame-related angular velocity *relative to the inertial frame*. Accordingly, the expressions $I_r(\mathrm{g}(A))$ and $h_r(\mathrm{g}(A))$ return the direction angles (as seen by an observer located on the body) of the instantaneous angular velocity relative to the *inertial frame*, i.e., the velocity observed in [43]. This way, what might have been a problem of the canonical Kinoshita theory became its advantage. It is exactly due to the nonosculation of the Andoyer elements, introduced in a precessing ecliptic frame, that this theory always returns the angular velocity of the Earth relative to inertial axes. We see that sometimes loss of osculation may be an advantage of the theory.

4.4.6 Example 2. The theory of Mars rotation.

Two groups ([14], [15, 16]) independently investigated the long-term evolution of Mars' rotation, using a simplified and averaged version of the Kinoshita Hamiltonian. The goal was to obtain a long-term history of the Martian spin axis' obliquity, i.e., of the angle between the Martian spin axis and a normal to the Martian ecliptic. Just as in the afore described case of the Earth, the Martian spin axis is evolving due to the solar torque acting on the oblate Mars, while the Martian ecliptic plane is in precession due to the perturbations exerted upon Mars by the other planets. Since for realistic rotators the Andoyer angle J is typically very small (i.e., since the angular-velocity and angular-momentum vectors are almost parallel), one may, in astronomical applications, approximate the obliquity with the angle made by the planet's angular-momentum vector and the ecliptic. When the Andoyer construction is built in a precessing frame (the fiducial basis $(\hat{\mathbf{s}}_1\,,\,\hat{\mathbf{s}}_2\,,\,\hat{\mathbf{s}}_3)$ being fixed on the planet's ecliptic), this assertion means that the obliquity is approximated with the Andoyer angle I . When the Andoyer variables are introduced in the traditional way (the basis $(\hat{\mathbf{s}}_1\,,\,\hat{\mathbf{s}}_2\,,\,\hat{\mathbf{s}}_3)$ being inertial), the above assertion means that one has to find the orientation of the angular momentum relative to the the inertial axes (i.e., to calculate the Euler angles h , I) and to find the orientation of the ecliptic relative to the same inertial axes. Then the orientation of the angular momentum with respect to the ecliptic will be found, and it will be an approximation for the obliquity.

The former approach was implemented by Laskar and Robutel [14], the latter by Touma and Wisdom [15, 16]. Despite the fact that these two teams introduced the Andoyer variables in different frames of reference, the outcomes of their calculations were very close, minor differences being attributed to other reasons.[13] This coincidence stems from the afore explained fact that the Andoyer variables and the Andoyer angles I, J retain their physical meaning when introduced in a precessing frame. (See subsection 4.4.3.)

5 The Sadov Variables

The Andoyer variables have the merit that they reduce the Hamiltonian of an unsupported and torque-free rigid body to one and a half degrees of freedom, the resulting expression for the Hamiltonian being very simple. However, except for the case of axial symmetry, these variables are not action–angle.

Sadov [25] and Kinoshita [23], in an independent way, obtained sets of action–angle variables for the rotational motion of a triaxial rigid body. Both sets of variables are very similar. Essentially, they are obtained by solving the Hamilton-Jacobi equation stemming from the Hamiltonian in the Andoyer variables. Sadov's transformation is formulated in terms of the Legendre elliptic functions of the first and third kind, while in the Kinoshita transformation the Heuman Lambda function emerges.

Here we present the approach of Sadov who began with introducing an intermediate set of canonical variables $(\beta\,,\ \alpha)$ that nullify the Hamiltonian. The generating function $\mathcal{S}$ of the transformation

$$(\ell, g, h, L, G, H) \longrightarrow (\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3),$$

can be found through solving the corresponding Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\sin^2\ell}{I_1}+\frac{\cos^2\ell}{I_2}\right)\left[\left(\frac{\partial\mathcal{S}}{\partial g}\right)^2-\left(\frac{\partial\mathcal{S}}{\partial\ell}\right)^2\right]+\frac{1}{2I_3}\left(\frac{\partial\mathcal{S}}{\partial\ell}\right)^2+\frac{\partial\mathcal{S}}{\partial t}=0 \quad .$$

Since the system is autonomous and variables g and h are cyclic, the generating function may be expressed as

$$\mathcal{S}=-\alpha_1\,t+\alpha_2 g+\alpha_3 h+\mathcal{U}(\ell;\alpha_1,\alpha_2,\alpha_3) \quad ,$$

[13] While Touma and Wisdom [15, 16] employed unaveraged equations of motion, Laskar and Robutel [14] used orbit-averaged equations.

wherefrom

$$\left(\frac{\partial\mathcal{U}}{\partial\ell}\right)^2\left[\left(\frac{1}{I_3}-\frac{1}{I_2}\right)-\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\sin^2\ell\right]=\left(2\alpha_1^2-\alpha_2^2\frac{1}{I_2}\right)-\alpha_2^2\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\sin^2\ell \quad ,$$

further simplification whereof yields a simple equation for the function $\mathcal{U}$:

$$\left(\frac{\partial\mathcal{U}}{\partial\ell}\right)^2=I_3\left(\frac{a+b\sin^2\ell}{c+d\sin^2\ell}\right),$$

with

$$a=I_1(\alpha_2^2-2\alpha_1 I_2)\quad ,\qquad c=I_1(I_3-I_2)>0\quad ,$$
$$b=(I_2-I_1)\alpha_2^2>0\quad ,\qquad d=I_3(I_2-I_1)>0\quad .$$

Hence, the transformation is

$$\begin{aligned} L&=\sqrt{I_3}\,\sqrt{\frac{a+b\sin^2\ell}{c+d\sin^2\ell}}\quad , & \beta_1&=-t+\frac{\partial\mathcal{U}}{\partial\alpha_1}\quad ,\\ G&=\frac{\partial\mathcal{S}}{\partial g}=\alpha_2\quad , & \beta_2&=g+\frac{\partial\mathcal{U}}{\partial\alpha_2}\quad ,\\ H&=\frac{\partial\mathcal{S}}{\partial h}=\alpha_3\quad , & \beta_3&=h\quad , \end{aligned} \tag{135}$$

whence we see that

$$\mathcal{H}=-\frac{\partial\mathcal{S}}{\partial t}=\alpha_1. \tag{136}$$

At this point, Sadov introduced[14] a parameter κ and a so-called state function λ as

$$\kappa^2=\frac{I_3(I_2-I_1)}{I_1(I_3-I_2)}\geq 0\quad ,\qquad \lambda^2=\kappa^2\,\frac{I_1}{I_3}\,\frac{2\,I_3\alpha_1-\alpha_2^2}{\alpha_2^2-2\,I_1\alpha_1}\geq 0\quad . \tag{137}$$

Via these quantities, the angular momentum L in (135) may be expressed as

$$L^2=\alpha_2^2\,\frac{\kappa^2}{\kappa^2+\lambda^2}\,\frac{1-\lambda^2+(\kappa^2+\lambda^2)\sin^2\ell}{1+\kappa^2\sin^2\ell}\quad , \tag{138}$$

[14]These quantities already appear, in a different context, in [21], p. 394

and the Hamiltonian (136) becomes:

$$\mathcal{H} = \frac{\alpha_2^2}{2\,I_1 I_3}\,\frac{I_3\lambda^2 + I_1\kappa^2}{\kappa^2+\lambda^2}.$$

With this transformation accomplished, Sadov proceeded to a second one,

$$(\ell, g, h, L, G, H) \longrightarrow (\varphi_\ell, \varphi_g, \varphi_h, I_\ell, I_g, I_h) \quad , \tag{139}$$

the new actions being

$$I_\ell = \frac{1}{2\pi}\oint L\,d\ell, \qquad I_g = \frac{1}{2\pi}\oint G\,dg = G = \alpha_2, \qquad I_h = \frac{1}{2\pi}\oint H\,dh = H = \alpha_3 \quad .$$

While the variables $(\beta\,,\,\alpha)$ corresponded to a vanishing Hamiltonian, the action–angle ones correspond to the initial Hamiltonian of Andoyer (though now this Hamiltonian has, of course, to be expressed through these new variables).

By means of a convenient auxiliary variable z introduced through

$$\sin\ell = \frac{\cos z}{\sqrt{1+\kappa^2\sin^2 z}}, \qquad \cos\ell = -\frac{\sqrt{1+\kappa^2}\sin z}{\sqrt{1+\kappa^2\sin^2 z}}, \tag{140}$$

it is possible to derive from (138) that

$$I_\ell = \frac{1}{2\pi}\oint L\,d\ell = \frac{2\alpha_2\sqrt{1+\kappa^2}}{\pi\kappa\sqrt{\kappa^2+\lambda^2}}\left[(\kappa^2+\lambda^2)\Pi(\kappa^2,\lambda) - \lambda^2\,K(\lambda)\right]. \tag{141}$$

where K and Π are the complete elliptical integrals of the first and the third kind, respectively.

Symbolically, the above expression may be written as $I_\ell = I_\ell(I_g\,,\,\lambda) = I_g\,f(\lambda)$. According to the Implicit Function Theorem, there exists a function $\phi(I_\ell/I_g)$ inverse, locally, to $f(\lambda)$. Let us denote this function as $\lambda = \phi(I_\ell/I_g)$. Sadov [25] proved that it is defined for all values $(I_\ell/I_g) \geq 0$, and that it is analytic for $(I_\ell/I_g) > 0$ or $(I_\ell/I_g) \neq (2/\pi)\arctan\kappa$ at $\lambda = 1$.

Since the Hamiltonian is known to be

$$\mathcal{H} = \frac{I_g^2}{2\,I_1 I_3}\,\frac{I_3\lambda^2 + I_1\kappa^2}{\kappa^2+\lambda^2}, \tag{142}$$

the angular variables could be found by integrating the Hamilton equations

$$\dot{\varphi}_\ell = \frac{\partial\mathcal{H}}{\partial I_\ell}\,, \qquad \dot{\varphi}_g = \frac{\partial\mathcal{H}}{\partial I_g}\,, \qquad \dot{\varphi}_h = \frac{\partial\mathcal{H}}{\partial I_h} \quad ,$$

if the explicit relation between the action momenta and λ were known. Unfortunately we have not such relation at hand. To circumvent this difficulty, the generating function of the transformation must be derived.

Let $\mathcal{W}$ be the generating function of the transformation (139) from the Andoyer variables to the action and angle variables. It may be chosen as

$$\mathcal{W} = I_g g + I_h h + \mathcal{V}(\ell; I_\ell, I_g) \quad ,$$

and the equations of this transformation will read:

$$\begin{aligned} L &= \frac{\partial \mathcal{V}}{\partial \ell} \quad , & \varphi_\ell &= \frac{\partial \mathcal{V}}{\partial I_\ell} \quad , \\ G &= \frac{\partial \mathcal{W}}{\partial g} = I_g \quad , & \varphi_g &= \frac{\partial \mathcal{W}}{\partial I_g} = g + \frac{\partial \mathcal{V}}{\partial I_g} \quad , \\ H &= \frac{\partial \mathcal{W}}{\partial h} = I_h \quad , & \varphi_g &= \frac{\partial \mathcal{W}}{\partial I_h} = h \quad . \end{aligned}$$

The generating function $\mathcal{V}$ is obtained directly from the quadrature

$$\mathcal{V}(\ell; I_\ell, I_g) = \int_{\ell_0}^{\ell} L(\ell; I_\ell, I_g)\, d\ell \quad ,$$

where we have to replace L with expression (138).

To calculate φ_ℓ and φ_g, we need to find the partial derivatives $\partial\lambda/\partial I_\ell$ and $\partial\lambda/\partial I_g$. Since $\lambda = \phi(I_\ell/I_g)$ and $I_\ell = I_g f(\lambda)$, then

$$\frac{\partial I_\ell}{\partial \lambda} = I_g \frac{\partial f}{\partial \lambda} = \frac{1}{2\pi} \oint \frac{\partial L}{\partial \lambda} = -\frac{I_g 2\kappa\lambda(1+\kappa^2)^{1/2}}{\pi(\kappa^2+\lambda^2)^{3/2}}\, K(\lambda)$$

After introducing notation $u \equiv I_\ell/I_g$, we can write:

$$\frac{\partial \phi}{\partial u} = \frac{1}{\partial u/\partial \phi} = I_g \frac{1}{\partial I_\ell/\partial \phi} = I_g \frac{1}{\partial I_\ell/\partial \lambda} = -\frac{\pi(\kappa^2+\lambda^2)^{3/2}}{2\kappa\lambda(1+\kappa^2)^{1/2}K(\lambda)} \tag{143}$$

whence

$$\frac{\partial \lambda}{\partial I_g} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial I_g} = \frac{I_\ell}{I_g^2}\frac{\pi(\kappa^2+\lambda^2)^{3/2}}{2\kappa\lambda(1+\kappa^2)^{1/2}K(\lambda)} = \frac{(\kappa^2+\lambda^2)}{I_g\kappa^2\lambda K(\lambda)}\left[(\kappa^2+\lambda^2)\Pi(\kappa^2,\lambda) - \lambda^2 K(\lambda)\right].$$

Now, taking into account (138) and employing the chain rule, we arrive at

$$\begin{aligned} \varphi_\ell &= \frac{\partial \mathcal{V}}{\partial I_\ell} = \frac{\partial \mathcal{V}}{\partial \lambda}\frac{\partial \lambda}{\partial I_\ell} = \frac{\partial \lambda}{\partial I_\ell}\int_{\ell_0}^{\ell}\frac{\partial L}{\partial \lambda} d\ell \quad , \\ \varphi_g &= g + \frac{\partial}{\partial I_g}\int_{\ell_0}^{\ell} L\, d\ell = g + \frac{1}{I_g}\int_{\ell_0}^{\ell} L\, d\ell + \frac{\partial \lambda}{\partial I_g}\int_{\ell_0}^{\ell}\frac{\partial L}{\partial \lambda}\, d\ell \quad . \end{aligned} \tag{144}$$

Straightforward differentiation of (138) yields:

$$\frac{\partial L}{\partial \lambda} = -\frac{I_g \kappa \lambda (1+\kappa^2)}{(\kappa^2+\lambda^2)^{3/2}} \frac{1}{\sqrt{1+\kappa^2 \sin^2 \ell}\sqrt{1-\lambda^2+(\kappa^2+\lambda^2)\sin^2 \ell}}.$$

With the help of (140), we can now compute the quadratures

$$\int_{\ell_0}^{\ell} \frac{\partial L}{\partial \lambda}\, d\ell = -\frac{I_g \kappa \lambda (1+\kappa^2)^{1/2}}{(\kappa^2+\lambda^2)^{3/2}}\, F(z,\lambda)$$

and

$$\int_{\ell_0}^{\ell} L\, d\ell = \frac{I_g}{\kappa}\sqrt{\frac{1+\kappa^2}{\kappa^2+\lambda^2}}\Big[(\kappa^2+\lambda^2)\Pi(z,\kappa^2,\lambda) - \lambda^2 K(\lambda)\Big] \quad ,$$

insertion whereof into Eqs. (144) results in

$$\begin{aligned} \varphi_\ell &= \frac{\pi}{2}\frac{F(z,\lambda)}{K(\lambda)}, \\ \varphi_g &= g + \frac{1}{\kappa}\sqrt{(\kappa^2+\lambda^2)(1+\kappa^2)}\left[\Pi(z,\kappa^2,\lambda) - \frac{\Pi(\kappa^2,\lambda)F(z,\lambda)}{K(\lambda)}\right] \quad . \end{aligned} \tag{145}$$

Inversion of the first of these expressions entails

$$z = \mathrm{am}(2\varphi_\ell K(\lambda)/\pi,\, \lambda) \quad .$$

(For the definition of the elliptic function *am* see Appendix A2 below.) Thus, we have the Andoyer angle g expressed via the action-angle variables:

$$g = \varphi_g + \frac{1}{\kappa}\sqrt{(\kappa^2+\lambda^2)(1+\kappa^2)}\left[\frac{2}{\pi}\Pi(\kappa^2,\lambda)\,\varphi_\ell - \Pi\Big(\mathrm{am}(2\varphi_\ell K(\lambda)/\pi\,,\,\lambda),\kappa^2,\lambda\Big)\right] \quad ,$$

while the angle ℓ is obtained directly from expression (140).

6 CONCLUSIONS

In this paper, we have reviewed the Serret-Andoyer (SA) formalism for modeling and control of rigid-body dynamics from the dynamical systems perspective. We have dwelt upon the important topic of modeling of general, possibly angular velocity-dependant disturbing torques, and upon the interconnection between the Andoyer and the Sadov sets of variables. We have also contributed some new insights.

The first insight is that the Andoyer variables turn out to be non-osculating in the general case of angular-velocity-dependent perturbation. The second insight is that even when these variables are introduced in a precessing reference frame, they preserve their interconnection with the components of the angular momentum – a circumstance that makes the Andoyer variables especially valuable in astronomical calculations.

In summary, this treatise constitutes a first step towards understanding the consequences of using the SA formalism as a single, generic language for modelling rigid body dynamics in diverse – and seemingly unrelated – areas such as celestial mechanics, satellite attitude control, and geometric mechanics.

Appendix.

A.1. Spherical-trigonometry formula (57)

The standard formula of the spherical trigonometry [30],

$$\cos g \;=\; \cos(\phi - h)\,\cos(\psi - l) \,+\, \sin(\phi - h)\,\sin(\psi - l)\,\cos(\pi - \theta) \quad , \tag{146}$$

immediately entails:

$$\begin{aligned}\sin g\; dg \;=\; &\left[\,\sin(\phi - h)\;\cos(\psi - l) \;-\; \sin(\phi - h)\;\sin(\psi - l)\;\cos\theta\,\right] d(\phi - h)\\
&+ \left[\,\cos(\phi - h)\;\sin(\psi - l) \;-\; \sin(\phi - h)\;\cos(\psi - l)\;\cos\theta\,\right] d(\psi - l)\\
&\qquad +\; \sin(\phi - l)\;\;\sin(\psi - l)\;\;\sin\theta\;\;d\theta \quad .\end{aligned} \tag{147}$$

The other standard formulae of the spherical trigonometry enable one to transform (147) into

$$\begin{aligned}&\sin g\; dg \;=\\
&\sin g\;\;\cos I\;\,d(\phi - h) \;+\; \sin g\;\;\cos J\;\,d(\psi - l) \;+\; \sin(\phi - h)\;\;\sin(\psi - l)\;\;\sin\theta\;d\theta \;=\\
&\sin g\;\;\cos I\;\,d(\phi - h) \;+\; \sin g\;\;\cos J\;\,d(\psi - l) \;+\; \sin(\psi - l)\;\;\sin J\;\;\sin g\;d\theta \quad ,\end{aligned} \tag{148}$$

wherefrom equality (57) follows.

A.2. The dimensionless time u and its interconnection with T_{kin} , G , $G\cos I$, θ , and ψ

First let us recall some basics. As agreed in the text, we choose the body axes to coincide with the principal axes of inertia, $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, $\hat{\mathbf{b}}_3$. This makes the inertia tensor look like:

$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3 \end{pmatrix} \quad , \tag{149}$$

Decomposing the body angular velocity $\boldsymbol{\omega}$ over the principal basis,

$$\boldsymbol{\omega} \;=\; \omega_1\, \hat{\mathbf{b}}_1 \;+\; \omega_2\, \hat{\mathbf{b}}_2 \;+\; \omega_3\, \hat{\mathbf{b}}_3 \quad , \tag{150}$$

we can write twice the kinetic energy as

$$2\, T_{kin} \;=\; \boldsymbol{\omega}\, \mathbb{I}\, \boldsymbol{\omega} \;=\; I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \quad , \tag{151}$$

and the body-frame-related angular-momentum vector as

$$\mathbf{g} = \mathbb{I} \cdot \boldsymbol{\omega} \quad . \tag{152}$$

The square of this vector will be:

$$\mathbf{g}^2 \;=\; I_1^2\, \omega_1^2 \;+\; I_2^2\, \omega_2^2 + I_3^2\, \omega_3^2 \quad , \tag{153}$$

while its direction cosines with respect to an invariable plane-based coordinate system will read:[15]

$$\alpha' \;=\; \frac{\mathrm{g}_1}{G} \;=\; \frac{I_1\, \omega_1}{G} \quad , \qquad \beta' \;=\; \frac{\mathrm{g}_2}{G} \;=\; \frac{I_2\, \omega_2}{G} \quad , \qquad \gamma' \;=\; \frac{\mathrm{g}_3}{G} \;=\; \frac{I_3\, \omega_3}{G} \quad , \tag{154}$$

G denoting the magnitude of the angular momentum vector: $G \equiv |\mathbf{g}|$. At this point it is convenient to define an auxiliary quantity P^2 via

$$P^2 \;\equiv\; \frac{(\alpha')^2}{I_1} \;+\; \frac{(\beta')^2}{I_2} \;+\; \frac{(\gamma')^2}{I_3} \quad . \tag{155}$$

Substitution of (154) into (155) shows that this auxiliary quantity obeys

$$G^2\, P^2 \;=\; I_1\, \omega_1^2 \;+\; I_2\, \omega_2^2 \;+\; I_3\, \omega_3^2 \tag{156}$$

[15] The direction cosines can be expressed in terms of the angle l from the ascending node of the equator on the invariable plane to the the $\hat{\mathbf{b}}_1$ body axis, and the angle J between the angular-momentum vector and the $\hat{\mathbf{b}}_3$ principal axis of the body (Fig. 2):

$$\alpha' \;=\; \sin J\; \sin l \quad , \qquad \beta' \;=\; \sin J\; \cos l \qquad \gamma' \;=\; \cos J \quad ,$$

so that

$$I_1\, \omega_1 \;=\; G\; \sin J\; \sin l \quad , \qquad I_2\, \omega_2 \;=\; G\; \sin J\; \cos l \quad , \qquad I_3\, \omega_3 \;=\; G\; \cos J \quad .$$

or, equivalently,

$$G^2 \, P^2 \; = \; 2 \, T_{kin} \quad . \tag{157}$$

We see that P^2 is an integral of motion, a circumstance that will later help us with reduction of the problem.

Due to the evident identity

$$(\alpha')^2 + (\beta^{'})^2 + (\gamma^{'})^2 = 1 \tag{158}$$

our P^2 depends upon only two directional cosines. Elimination of $\gamma^{'}$ from (155), by means of (158), trivially yields:

$$P^2 - \frac{1}{I_3} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)(\alpha')^2 + \left(\frac{1}{I_2} - \frac{1}{I_3}\right)(\beta^{'})^2. \tag{159}$$

Note that, if we now introduce an a^2 and a b^2 such that

$$a^2 \; = \; \frac{P^2 \; - \; \frac{1}{I_3}}{\frac{1}{I_1} \; - \; \frac{1}{I_3}} \quad , \tag{160}$$

$$b^2 \; = \; \frac{P^2 \; - \; \frac{1}{I_3}}{\frac{1}{I_2} \; - \; \frac{1}{I_3}} \quad , \tag{161}$$

we shall be able to write down the definition of P^2 in a form that will, formally, be identical to definition of an ellipse:

$$\frac{(\alpha')^2}{a^2} + \frac{(\beta^{'})^2}{b^2} = 1 \quad . \tag{162}$$

This means that, along with P^2 , it is convenient to introduce another auxiliary variable, ξ , one that obeys

$$\frac{(\alpha')^2}{a^2} \; \equiv \; \cos^2 \xi \quad , \qquad \frac{(\beta^{'})^2}{b^2} \; \equiv \; \sin^2 \xi \quad . \tag{163}$$

Insertion of the above into (158) entails

$$(\gamma^{'})^2 \; = \; \left(1 \; - \; a^2\right) \left(1 \; - \; \frac{b^2 \; - \; a^2}{1 \; - \; a^2} \, \sin^2 \xi\right) \quad . \tag{164}$$

Then, after defining the quantity κ^2 and the function Δ as

$$\kappa^2 \equiv \frac{b^2 - a^2}{1 - a^2} \tag{165}$$

and

$$\Delta\xi \equiv \sqrt{1 - \kappa^2 \sin^2 \xi} \quad , \tag{166}$$

we shall be able to cast the direction cosines of the angular-momentum vector in the form of

$$\alpha' = a \cos \xi \quad , \tag{167}$$

$$\beta^{'} = b \sin \xi \quad , \tag{168}$$

$$\gamma^{'} = \sqrt{1 - a^2}\, \Delta\xi \quad . \tag{169}$$

Looking back at (154), we see that initially we started out with three direction cosines, only two of which were independent due to the equality (158). The latter meant that all these cosines might be expressed via two independent variables. The quantities P^2 and ξ were cast for the part. (Mind that a and b depend on P^2 through (160) - (161).) It should also be mentioned that, due to (158), P^2 is a constant of motion, and therefore formulae (167 - 169) effectively reduce the problem to one variable, ξ, which thereby plays the role of re-scaled time, in terms whereof the problem is fully solved. Below we shall explicitly write down the dependence of the "time" ξ upon the real time t or, equivalently, upon the dimensionless time $u \equiv n\,(t - t_o)$ emerging in (61 - 62).

Undisturbed spin of an unsupported rigid body obeys the Euler equations for the principal-axes-related components of the body angular velocity:

$$I_1 \frac{d\omega_1}{dt} = \omega_2\, \omega_3\, (I_2 - I_3) \quad , \tag{170}$$

$$I_2 \frac{d\omega_2}{dt} = \omega_3\, \omega_1\, (I_3 - I_1) \quad , \tag{171}$$

$$I_3 \frac{d\omega_3}{dt} = \omega_1\, \omega_2\, (I_1 - I_2) \quad . \tag{172}$$

Insertion of expressions (154) into (170), (171), and (172) provides an equivalent description written in terms of the angular-momentum vector's directional cosines relative to the principal axes:

$$\frac{d\alpha'}{dt} \;=\; G\,\beta^{'}\,\gamma^{'}\,\left(\frac{1}{I_3}\;-\;\frac{1}{I_2}\right)\quad , \tag{173}$$

$$\frac{d\beta^{'}}{dt} \;=\; G\,\gamma^{'}\,\alpha'\,\left(\frac{1}{I_1}\;-\;\frac{1}{I_3}\right)\quad , \tag{174}$$

$$\frac{d\gamma^{'}}{dt} \;=\; G\,\alpha'\,\beta^{'}\,\left(\frac{1}{I_2}\;-\;\frac{1}{I_1}\right)\quad . \tag{175}$$

Substitution of (167 - 169) into (173) will then entail

$$\frac{d\xi}{\Delta\xi} \;=\; \frac{b}{a}\,\sqrt{1\;-\;a^2}\,\left(\frac{1}{I_2}\;-\;\frac{1}{I_3}\right)\;G\;dt\quad . \tag{176}$$

A subsequent insertion of (160 - 161) and (166) into (176) will then entail:

$$\frac{d\xi}{\sqrt{1\;-\;\kappa^2\;\sin^2\xi}} \;=\; G\;dt\,\sqrt{\left(\frac{1}{I_1}\;-\;P^2\right)\left(\frac{1}{I_2}\;-\;\frac{1}{I_3}\right)}\quad . \tag{177}$$

Now define a "mean motion" n as

$$n\;\equiv\;G\,\sqrt{\left(\frac{1}{I_1}\;-\;P^2\right)\;\left(\frac{1}{I_2}\;-\;\frac{1}{I_3}\right)}\quad , \tag{178}$$

Since, according to (157), P^2 is integral of motion, then so is n , and therefore

$$\int_0^{\xi}\frac{d\xi}{\sqrt{1\;-\;\kappa^2\;\sin^2\xi}} \;=\; \int_{t_o}^{t} n\;dt' \;=\; n\,(t\;-\;t_o)\quad . \tag{179}$$

This means that, if we define a dimensionless time as

$$u\;\equiv\;\int_{t_o}^{t} n\;dt' \;=\; n\,(t\;-\;t_o) \tag{180}$$

and a function F as

$$F(\xi\,,\ \kappa)\ \equiv\ \int_0^{\xi}\frac{d\xi'}{\sqrt{1\ -\ \kappa^2\ \sin^2\xi'}}\quad ,\tag{181}$$

then the interrelation between the parameter ξ and the time will look like:

$$u\ =\ F(\xi\,,\ \kappa)\quad .\tag{182}$$

This is a Jacobi elliptic equation whose solutions are be written in terms of the following elliptic functions:

$$\xi\ =\ \mathrm{am}(u\,,\ \kappa)\ \equiv\ F^{-1}(u\,,\ \kappa)\quad ,\tag{183}$$

$$\sin\xi\ =\ \mathrm{sn}(u\,,\ \kappa)\ \equiv\ \sin(\,\mathrm{am}(u\,,\ \kappa)\,)\quad ,\tag{184}$$

$$\cos\xi\ =\ \mathrm{cn}(u\,,\ \kappa)\ \equiv\ \cos(\,\mathrm{am}(u\,,\ \kappa)\,)\quad ,\tag{185}$$

$$\Delta\xi\ \equiv\ \sqrt{1\ -\ \kappa^2\ \mathrm{sn}^2(u\,,\ \kappa)}\ \equiv\ \mathrm{dn}(u\,,\ \kappa)\quad .\tag{186}$$

According to (182), u is a function of ξ and κ . The integral of motion κ is, through (165) and (157), a function of T_{kin} and G , while the parameter ξ is, through the medium of (160 - 161) and (167 - 168), a function of T_{kin} , G , $\alpha' = \sin J\ \sin l$, and $\beta^{'}\ =\ \sin J\ \cos l$. All in all, u is a function of T_{kin} , G , J , and l .

It is also possible to express u through another set of geometric variables. Recall that $(\,\phi\,,\,\theta\,,\,\psi\,)$ are the Euler angles defining orientation of the body axes $(\,\hat{\mathbf{b}}_1\,,\,\hat{\mathbf{b}}_2\,,\,\hat{\mathbf{b}}_3\,)$ relative to the fiducial frame $(\,\hat{\mathbf{s}}_1\,,\,\hat{\mathbf{s}}_2\,,\,\hat{\mathbf{s}}_3\,)$, while $(\,\phi_0\,,\,J\,,\,l\,)$ are the Euler angles defining orientation of the body axes with respect to the invariable plane, as on (Fig. 2). Out of that picture, it is convenient to single out a spherical triangle whose sides are given by $(\,\phi\,-\,h\,,\,\psi\,-\,l\,,\,g\,)$, and whose internal angles are $(\,I\,,\,\pi\,-\,\theta\,,\,J\,)$, as shown in Fig. **??**. An analogue to the law of cosines, for spherical triangles, looks like:

$$\cos I\ =\ \cos\theta\ \cos J\ +\ \sin\theta\ \sin J\ \cos(\psi\,-\,l)\quad ,\tag{187}$$

which is the same as (146), up to a cyclic transposition. The second term on the right-hand side of (187) can be expanded as

$$
\begin{aligned}
\sin J \;\cos(\psi\,-\,l) \;&=\; \sin J\;\left(\cos\psi\;\cos l\,+\,\sin\psi\;\sin l\,\right)\\
&=\;\left(\,\sin J\;\sin l\,\right)\;\sin\psi\;+\;\left(\,\sin J\;\cos l\,\right)\;\cos\psi\\
&=\;\alpha'\;\sin\psi+\beta^{'}\;\cos\psi\\
&=\;a\;\sin\psi\;\cos(\,am(u)\,)\;+\;b\;\cos\psi\;\sin(\,am(u)\,)\quad ,
\end{aligned}
\tag{188}
$$

where the last line was obtained with aid of (167 - 168) and (183). Thus (187) acquires the shape of

$$
\begin{aligned}
\cos I \;=\; \cos\theta\,\sqrt{1\,-\,a^2}\;\Delta\mathrm{am}(u) \quad &+ \quad a\;\sin\theta\;\sin\psi\;\cos(\mathrm{am}(u))\\
&+ \quad b\;\cos\psi\;\sin(\mathrm{am}(u)) \quad .
\end{aligned}
\tag{189}
$$

We see that u depends upon ψ , θ , $\cos I$, and (through a and b) upon T_{kin} and G . This is equivalent to saying that u is a function of T_{kin} , G , $G\,\cos I$, θ , ψ .

A.3. Taking variations of S

Let us start with the expression (58) for the generating function:

$$
S \;=\; -\,t\,T_{kin}\;+\;G\;h\;\cos I\;+\;G\;g\;+\;G\;\int\cos J\;dl\;+\;C \quad . \tag{190}
$$

We see that it depends upon eight variables, some of which are dependent upon others. In brief,

$$
S \;=\; S(\,t\;,\;T_{kin}\;,\;l\;,\;L\equiv G\;\cos J\;,\;g\,,\;G\;,\;h\;,\;H\;\equiv\;G\;\cos I\,) \quad . \tag{191}
$$

Variation thereof, taken at a fixed time t , will look as:

$$
\delta S \;=\; \left(\frac{\partial S}{\partial T_{kin}}\right)\delta T_{kin}+\left(\frac{\partial S}{\partial h}\right)\delta h+\left(\frac{\partial S}{\partial (G\;\cos I)}\right)\delta(G\;\cos I)
$$

$$
+\left(\frac{\partial S}{\partial G}\right)\delta G+\left(\frac{\partial S}{\partial g}\right)\delta g+\left(\frac{\partial S}{\partial (G\;\cos J)}\right)\delta(G\;\cos J)+\left(\frac{\partial S}{\partial l}\right)\delta l
$$

$$= \; - \, t \, \delta T_{kin} \; + \; G \; \cos I \; \delta h \; + \; h \, \delta(G \; \cos I)$$

$$+ \left(g + \int \cos J \; dl \right) \delta G + G \; \delta g + G \int \delta(\cos J) \; dl \quad . \qquad (192)$$

Our goal is to simplify (192), having in mind that the variables emerging there are not all mutually independent. To that end, let us employ expression (148). Its differentiation would yield the expression

$$d \, g \, - \, d \, (\phi - h) \; \cos I \; = \; - \; \cos J \; d \, (l \, - \, \psi) + \sin J \; \sin \, (l - \psi) \; d\theta \quad , \qquad (193)$$

where the Euler angles ϕ, θ, ψ determine the orientation of the body relative to some inertial reference frame. If, however, we perform *variation* of (148), for a fixed orientation of the body relative to the inertial frame, then we shall get simply $\delta g \, = \, - \, \cos J \, \delta l \, - \, \cos I \, \delta h$. Multiplying this by G, we arrive at

$$G \; \delta g \; = \; - \, G \; \cos J \; \delta l \; - \; G \; \cos I \; \delta h \quad , \qquad (194)$$

substitution whereof into (192) entails

$$\delta S \; = \; - \, t \; \delta T_{kin} \; + \; h \; \delta(\, G \; \cos I \,) \; + \; \left(g \; + \; \int \cos J \; dl \right) \; \delta G$$

$$+ \; G \int \delta(\cos J) \; dl \; - \; G \; \cos J \; \delta l \quad . \qquad (195)$$

Next, we shall recall the formula for the Binet Ellipsoid, expressed through the directional cosines defined in (157. According to (155) and (157), we have:

$$\frac{(\alpha^{'})^2}{I_1} + \frac{(\beta^{'})^2}{I_2} + \frac{(\gamma^{'})^2}{I_3} \; = \; \frac{2 \; T_{kin}}{G^2} \quad . \qquad (196)$$

Insertion of the (evident from Fig. 2) relations

$$\alpha' \; = \; \sin J \; \sin l \quad , \qquad \beta^{'} \; = \; \sin J \; \cos l \qquad \gamma^{'} \; = \; \cos J \quad ,$$

into the above formula will yield

$$\sin^2 J \; \left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2} \right) + \frac{\cos^2 J}{I_3} \; = \; \frac{2 \; T_{kin}}{G^2} \quad , \qquad (197)$$

or, equivalently,

$$(1-(\gamma^{'})^2)\ \left(\frac{sin^2 l}{I_1}+\frac{cos^2 l}{I_2}-\frac{1}{I_3}\right)=\frac{2\ T_{kin}}{G^2}-\frac{1}{I_3}\quad . \tag{198}$$

Differentiation of the above will lead us to

$$2\ \frac{dT_{kin}}{G^2}\ -4\ \frac{T_{kin}\ dG}{G^3}\ =\ -\ 2\ \gamma^{'}\ d(\gamma^{'})\left(\frac{\sin^2 l}{I_1}+\frac{\cos^2 l}{I_2}-\frac{1}{I_3}\right)$$

$$+\ 2\ \alpha^{'}\beta^{'}\left(\frac{1}{I_1}-\frac{1}{I_2}\right)dl\quad . \tag{199}$$

If we now turn the differentials into variations, and multiply both sides by $G^2\ dt\ /\ 2$, we shall obtain:

$$\left(\delta T_{kin}\ -2\ \frac{T_{kin}\ \delta G}{G}\right)\ dt$$

$$=\ -G^2\ dt\ \gamma^{'}\ \delta\gamma^{'}\left(cos^2 l\left(\frac{1}{I_2}-\frac{1}{I_1}\right)+\left(\frac{1}{I_1}-\frac{1}{I_3}\right)\right)$$

$$+\ G^2\ dt\ \alpha^{'}\ \beta^{'}\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\delta l\quad . \tag{200}$$

Consider the second term on the right hand side of (200). We know from (175) that

$$G\ dt\ \alpha'\ \beta^{'}\ \left(\frac{1}{I_1}-\frac{1}{I_2}\right)\ =\ -\ d\gamma^{'}\quad , \tag{201}$$

so we can write the second term on the right hand side of (200) as

$$G^2\ dt\ \alpha^{'}\ \beta^{'}\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\delta l\ =\ -\ G\ d\gamma^{'}\ \delta l\quad . \tag{202}$$

The first term on the right hand side of (200) can be written down as

$$
\begin{aligned}
& - G^2\, dt\, \gamma^{'}\, \delta\gamma^{'} \left[\cos^2 l\ \left(\frac{1}{I_2} - \frac{1}{I_1}\right) + \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\right] \\
& \quad = \ - G^2\, dt\, \gamma^{'}\, \delta\gamma^{'} \left[\frac{(\beta^{'})^2}{1-(\gamma^{'})^2}\ \left(\frac{1}{I_2} - \frac{1}{I_1}\right) + \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\right] \\
& \quad = \ -G\, \gamma^{'}\, \delta\gamma^{'} \left(\frac{(\beta^{'})^2}{1-(\gamma^{'})^2}\ \frac{d\gamma^{'}}{\alpha^{'}\ \beta^{'}} + \frac{d\beta^{'}}{\alpha^{'}\ \gamma^{'}}\right) \\
& \quad = \ \frac{-G\, \delta\gamma^{'}}{\alpha^{'}}\ \left(\frac{\beta^{'}\ \gamma^{'}}{1-(\gamma^{'})^2}\ d\gamma^{'} + d\beta^{'}\right) \\
& \quad = \ \frac{-G\, \delta\gamma^{'}}{\alpha^{'}(1-(\gamma^{'})^2)}\ \left(\beta^{'}\ \gamma^{'}\ d\gamma^{'} + (1-(\gamma^{'})^2)\ d\beta^{'}\right) \quad . \qquad (203)
\end{aligned}
$$

We obtained the first equality by noting that the definitions of the direction cosines yield

$$\cos^2\ l\ =\ \frac{(\beta^{'})^2}{(\alpha^{'})^2 + (\beta^{'})^2}\ =\ \frac{(\beta^{'})^2}{1-(\gamma^{'})^2} \quad , \qquad (204)$$

while the second equality comes from the equations of motion (173 - 175) in Appendix A. Finally, when we look at $d\beta^{'}$, we see that

$$
\begin{aligned}
d\beta^{'}\ &=\ d(\sin\ J\ \cos\ l)\ =\ d(\sqrt{1-(\gamma^{'})^2}\ \cos\ l) \\
&=\ -\frac{\gamma^{'}\, d\gamma^{'}\ \cos l}{\sqrt{1-(\gamma^{'})^2}} - \sqrt{1-(\gamma^{'})^2}\ \sin l\, dl\ =\ -\frac{\gamma^{'}\, \beta^{'}\, d\gamma^{'}}{1-(\gamma^{'})^2} - \alpha^{'}\ dl \quad . \qquad (205)
\end{aligned}
$$

We can now write (203) as

$$\begin{aligned}
&\frac{-\,G\,\delta\gamma^{'}}{\alpha^{'}(1-(\gamma^{'})^2)}\,\left(\beta^{'}\,\gamma^{'}\,d\gamma^{'}+(1-(\gamma^{'})^2)\,d\beta^{'}\right)\\
&=\frac{-G\,\delta\gamma^{'}}{\alpha^{'}(1-(\gamma^{'})^2)}\,\left(\beta^{'}\,\gamma^{'}\,d\gamma^{'}+(1-(\gamma^{'})^2)\,\left(-\frac{\gamma^{'}\,\beta^{'}\,d\gamma^{'}}{1-(\gamma^{'})^2}-\alpha^{'}\,dl\right)\right)\\
&=\,G\,dl\,\delta\gamma^{'}\\
&=\,G\,\delta(\cos J)\,dl\quad .
\end{aligned}\tag{206}$$

This enables us to write (200) as

$$\left(\delta T_{kin}\,-2\,\frac{T_{kin}\,\delta G}{G}\right)\,dt=G\,\delta(\cos J)\,dl-G\,d(\cos J)\,\delta l\quad .\tag{207}$$

Integration of this yields

$$\begin{aligned}
G\int\delta(\cos J)\,dl\;&=\;\left(\delta T_{kin}\,-2\,\frac{T_{kin}\,\delta G}{G}\right)\,\int dt+G\,\int\delta l\,d\gamma^{'}\\
&=\;\left(\delta T_{kin}\,-2\,\frac{T_{kin}\,\delta G}{G}\right)\,\frac{u}{n}\;+\;G\;\cos J\,\delta l\quad .
\end{aligned}\tag{208}$$

Insertion of this result into (195) brings up the following

$$\delta S\;=\;\left(\frac{u}{n}\,-\,t\right)\;\delta T_{kin}\;+\;h\;\delta(G\;\cos I)\;+\;\left(g\;+\int\cos J\;dl\;-\;2\,\frac{T_{kin}}{G}\,\frac{u}{n}\right)\,\delta G\;\;.\tag{209}$$

Finally, getting rid of $\delta G\int\cos J\,dl$ by means of (61), we enjoy the desired formula (67).

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References

[1] Marsden, J. E., and Ratiu, T. S. 1999. *Introduction to Mechanics and Symmetry*, 2nd Ed., Springer, NY 1999.

[2] Deprit, A., and Elipe, A. 1993. "Complete Reduction of the Euler-Poinsot Problem," *The Journal of the Astronautical Sciences*, Vol. **41**, No. 4, pp. 603 - 628.

[3] Hori, Gen-ichiro. 1966. "Theory of General Perturbtions with Unspecified Canonical Variables." *Publications of the Astronomical Society of Japan.* Vol. **18**, pp. 287 - 296.

[4] Deprit, A. 1969. "Canonical Transformations Depending on a Small Parameter." *Celestial Mechanics*, Vol. **1**, pp. 12 - 30.

[5] Kholshevnikov, K. V. 1973. "Lie Transformations in celestial mechanics." In: *Astronomy* & *Geodesy.* Thematic collection of papers. Vol. **4**, Issue 4, pp. 21 - 45. Published by the Tomsk State University Press, Tomsk, Russia. /in Russian/

[6] Boccaletti, D., & G. Pucacco 2002. *Theory of Orbits. Volume 2: Perturbative and Geometrical Methods.* Springer Verlag, Heidelberg.

[7] Kholshevnikov, K. V. 1985. *Asymptotic Methods of Celestial Mechanics.* Leningrad State University Press, St.Petersburg, Russia, 1985. Chapter 5. /in Russian/

[8] Kinoshita, H. 1977. "Theory of the Rotation of the Rigid Earth," *Celestial Mechanics*, Vol. **15**, pp. 277 - 326.

[9] Escapa, A.; Getino, J.; & Ferrándiz, J. 2001. "Canonical approach to the free nutations of a three-layer Earth model." *Journal of Geophysical Research*, Vol.**106**, No B6, pp.11387-11397

[10] Escapa, A.; Getino, J.; & Ferrándiz, J. 2002. "Indirect effect of the triaxiality in the Hamiltonian theory for the rigid Earth nutations." *Astronomy & Astrophysics*, Vol. **389**,pp. 1047-1054

[11] Getino, J., & Ferrándiz, J. 1990."A Hamiltonian Theory for an Elastic Earth. Canonical Variables and Kinetic Energy." *Celestial Mechanics and Dynamical Astronomy*,Vol.**49**,pp.303-326

[12] Kinoshita, H., and Souchay, J. 1990. "The theory of the nutation for the rigid-Earth model at the second order." *Celestial Mechanics and Dynamical Astronomy*, Vol. **48**, pp. 187 - 265.

[13] Yoshida, H. 1993. "Recent Progress in the Theory and Applications of Symplectic Integrators." *Celestial Mechanics and Dynamical Astronomy*, Vol. **56**, pp. 27 - 43.

[14] Laskar, J., and Robutel, J. 1993. "The Chaotic Obliquity of the Planets." *Nature*, Vol. **361**, pp. 608 - 612.

[15] Touma, J., and J. Wisdom. 1993. "The Chaotic Obliquity of Mars." *Science.* Vol. **259**, No 5099, pp. 1294 - 1297.

[16] Touma, J., and Wisdom, J. 1994. "Lie-Poisson Integrators for Rigid Body Dynamics in the Solar System." *The Astronomical Journal*, Vol. **107**, pp. 1189 - 1202.

[17] Andoyer, H. 1923. *Cours de Mécanique Céleste*, Paris. Gauthier-Villars 1923.

[18] Deprit, A. 1967. "Free Rotation of a Rigid Body Studied in the Phase Plane," *American Journal of Physics*, Vol. **5**, pp. 424 - 428.

[19] Serret, J. A. 1866. "Mémoire sur l'emploi de la méthode de la variation des arbitraires dans théorie des mouvements de rotations." *Mémoires de l'Académie des sciences de Paris*, Vol. **35**, pp. 585 - 616.

[20] Radau, R. 1869. "Rotation des corps solides," *Annales de l'Ecole Normale Supérieure*, 1re série, 1869, pp. 585 - 616.

[21] Tisserand, F. 1889. *Traité de mécanique céleste*, tome II, pp. 382 - 393. Paris. Gauthier-Villars 1889.

[22] Richelot, M. 1850. "Eine neue Lœsung des Problems der Rotation." *Mémoires de l'Academie de Berlin*, pp. 1 - 59.

[23] Kinoshita, H. 1972. "First-Order Perturbations of the Two Finite-Body Problem," *Publications of the Astronomical Society of Japan*, Vol. **24**, pp. 423 - 457.

[24] Kane, T. R., Likins, P. W., and Levinson, D. A. 1983. *Spacecraft Dynamics*. McGraw Hill. New York 1983.

[25] Sadov, Iu. A. 1970. "The Action-Angle Variables in the Euler-Poinsot Problem", *Akad. Nauka SSSR*, Prep. 22.

[26] Schaub, H. and Junkins, J. L. 2003. *Analytical Mechanics of Space Systems.* AIAA Educational Series. AIAA. Reston VA, 2003.

[27] Giacaglia, G. E. O., and Jefferys, W. H. 1971. "Motion of a Space Station. I." *Celestial Mechanics*, Vol. **4**, pp. 442 - 467.

[28] Lum, K.-Y. and Bloch, A. M. 1999. "Generalized Serret-Andoyer Transformation and Applications for the Controlled Rigid Body," *Dynamics and Control*, Vol. **9**, pp. 39 - 66.

[29] Jacobi, C. G. J. 1866. *Vorlesungen über Dynamik*. G. Reimer, Berlin.

[30] Smart, W. M. 1965. *Text-Book on Spherical Astronomy.* Cambridge University Press. 1965.

[31] Leubner, C. 1981. "Correcting a wide-spread error concerning the angular velocity of a rotating body." *American Journal of Physics*, Vol. **49**, pp. 232 - 234.

[32] Efroimsky, M. 2005. "The theory of canonical perturbations applied to attitude dynamics and to the Earth rotation." Submitted to *Celestial Mechanics and Dynamical Astronomy.* astro-ph/0506427

[33] Plummer, H. C. 1918. *An Introductory Treatise on Dynamical Astronomy.* Cambridge University Press, UK.

[34] Fukushima, T., and Ishizaki, H. 1994. "Elements of Spin Motion." *Celestial Mechanics and Dynamical Astronomy*, Vol. **59**, pp. 149 - 159.

[35] Efroimsky, M., and Goldreich, P. 2003. "Gauge Symmetry of the N-body Problem in the Hamilton-Jacobi Approach." *Journal of Mathematical Physics*, Vol. **44**, pp. 5958 - 5977
astro-ph/0305344

[36] Efroimsky, M., & Goldreich, P. 2004. "Gauge Freedom in the N-body Problem of Celestial Mechanics." *Astronomy & Astrophysics*, Vol. **415**, pp. 1187 - 1199
astro-ph/0307130

[37] Efroimsky, M. 2006. "Gauge Freedom in Orbital Mechanics." *Annals of the New York Academy of Sciences*. Vol. **1065**, pp. 346 - 374.
astro-ph/0603092

[38] Murray, B. C.; Ward, W. R.; & Yeung, S. C. 1973. "Periodic Insolation Variations on Mars." *Science*, Vol. 180, pp. 638 - .

[39] Ward, W. 1973. "Large-scale Variations in the Obliquity of Mars." *Science*. Vol. **181**, pp. 260 - 262.

[40] Ward, W. 1974. "Climatic Variations of Mars. Astronomical Theory of Insolation." *Journal of Geophysical Research*, Vol. **79**, pp. 3375 - 3386.

[41] Efroimsky, M. 2005. "Long-term evolution of orbits about a precessing oblate planet. 1. The case of uniform precession." *Celestial Mechanics and Dynamical Astronomy*. Vol. **91**, pp. 75 - 108.
astro-ph/0408168

[42] Kinoshita, H.; Nakajima, K.; Kubo, Y.; Nakagawa, I.; Sasao, T.; & Yokoyama, K. 1978. "Note on Nutation in Ephemerides." *Publications of the International Latitude Observatory of Mizusawa*, Vol. **XII**, No 1, pp. 71 - 108.

[43] Schreiber, K. U.; Velikoseltsev, A,; Rothacher, M.; Klügel, T.; Stedman, G. E.; and Wiltshire, D. L. 2004. "Direct measurements of diurnal polar motion by ring laser gyroscopes." *Journal of Geophysical Research*, Vol. **109**, pp. B06405

[44] Brumberg, V. A., L. S. Evdokimova, & N. G. Kochina. 1971. "Analytical Methods for the Orbits of Artificial Satellites of the Moon." *Celestial Mechanics*, Vol. **3**, pp. 197 - 221.

[45] Brumberg, V.A. 1992. *Essential Relativistic Celestial Mechanics.* Adam Hilger, Bristol.

[46] Efroimsky, Michael. 2002a. "Equations for the orbital elements. Hidden symmetry." Preprint No 1844 of the Institute of Mathematics and its Applications, University of Minnesota
http://www.ima.umn.edu/preprints/feb02/feb02.html

[47] Efroimsky, Michael. 2002b. "The Implicit Gauge Symmetry Emerging in the N-body Problem of Celestial Mechanics."
astro-ph/0212245

[48] Goldreich, P. 1965. "Inclination of satellite orbits about an oblate precessing planet." *The Astronomical Journal*, Vol. **70**, pp. 5 - 9.

[49] Newman, W., & M. Efroimsky. 2003. "The Method of Variation of Constants and Multiple Time Scales in Orbital Mechanics." *Chaos*, Vol. 13, pp. 476 - 485.

[50] Bloch, A. M., Gurfil, P., and Lum, K.-Y.,"The Serret-Andoyer Formalism in Rigid-Body Dynamics: II. Geometry, Stabilization and Control", *Regular and Chaotic Dynamics*, submitted.

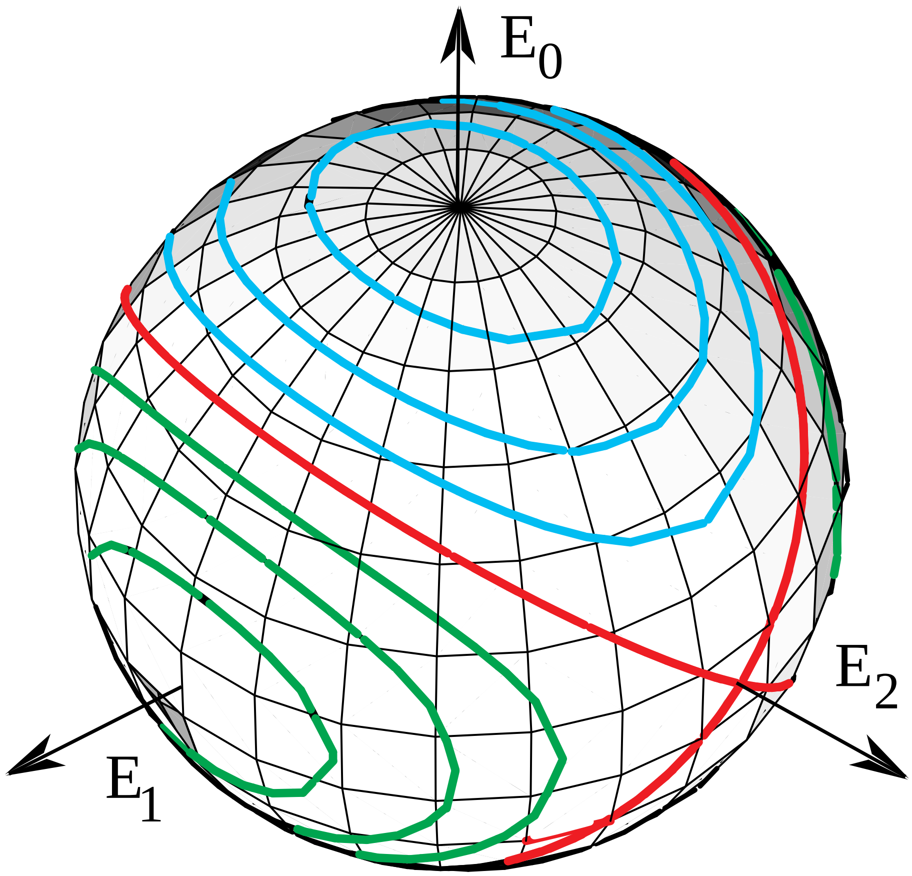

Figure 1: Phase flow of the Euler-Poinsot problem

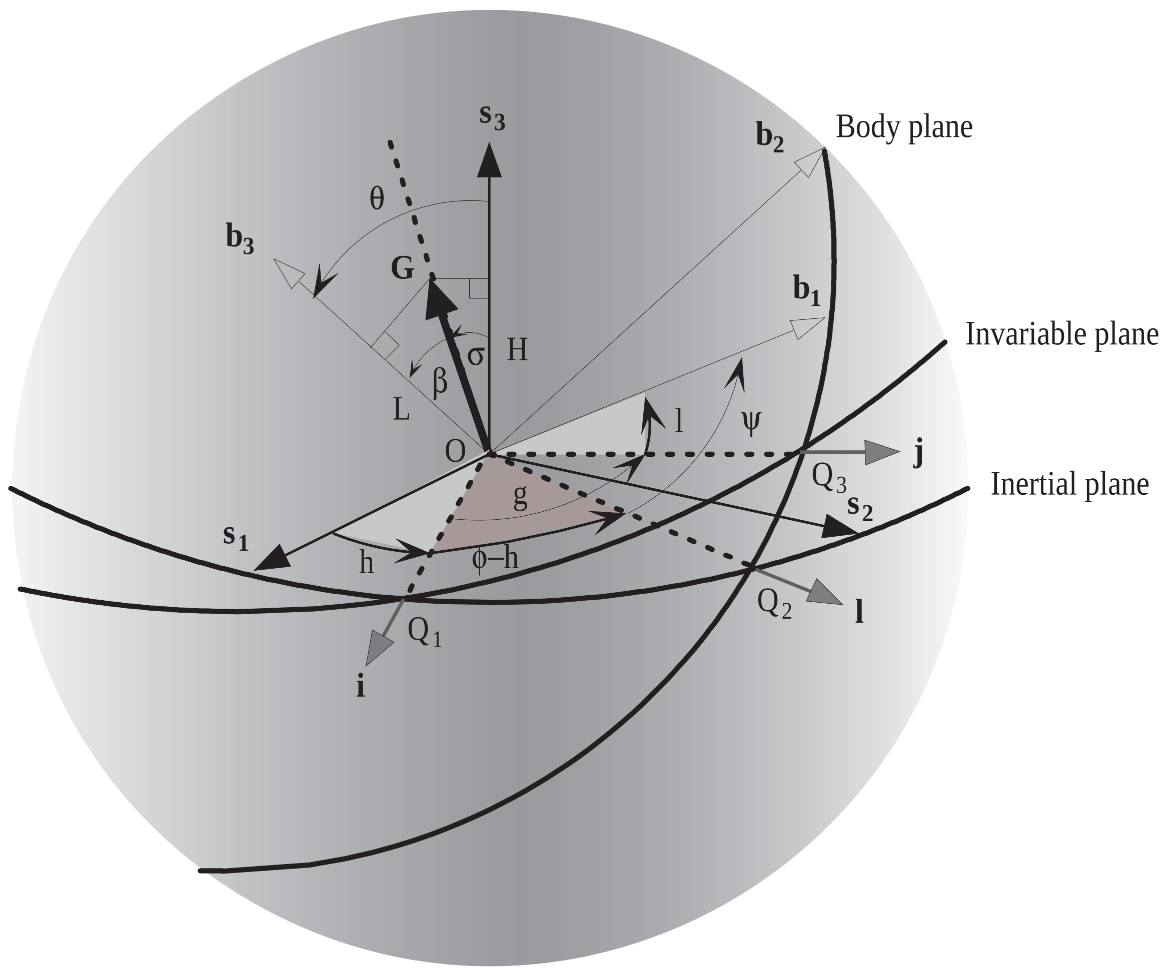

Figure 2: An inertial coordinate system, $\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_3$, a body-fixed frame, $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3$, an angular momentum-based frame and intersections of their fundamental planes, denoted by $\hat{\mathbf{i}}, \hat{\mathbf{l}}, \hat{\mathbf{j}}$.

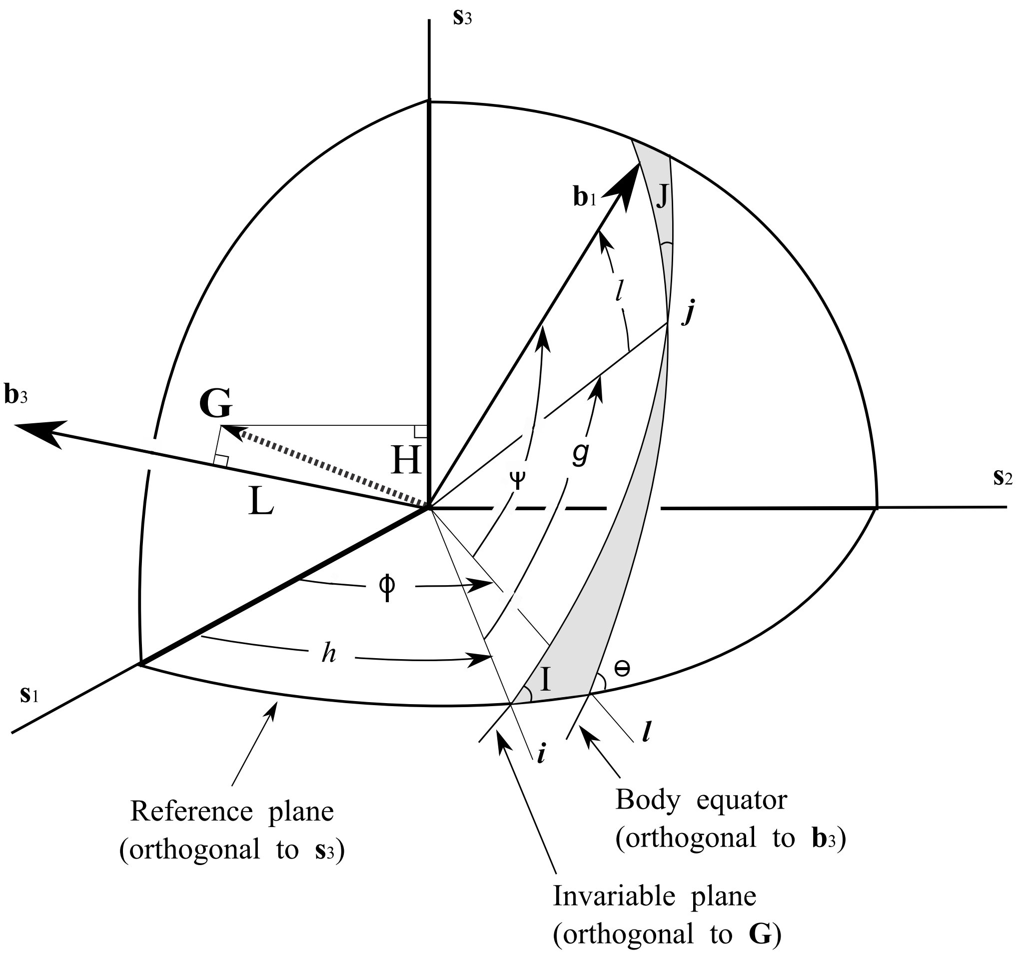

Figure 3: The same as the previous picture, but with fewer details.
The reference coordinate system (inertial or, more generally, precessing) is constituted by axes $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$. A body-fixed frame is defined by the principal axes $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. The third frame is constituted by the angular-momentum vector $\mathbf{G}$ and a plane orthogonal thereto (the so-called invariable plane). The lines of nodes are denoted with $\mathbf{i}$, $\mathbf{l}$, $\mathbf{j}$. The attitude of the body relative to the reference frame is given by the Euler angles ϕ , θ , ψ . The orientation of the invariable plane with respect to the reference frame is determined by the angles h and I . The inclination I is equal to the angle that the angular-momentum vector $\mathbf{G}$ makes with the reference axis $\mathbf{s}_3$. The angle J between the invariable plane and the body equator coincides with the angle that $\vec{\mathbf{G}}$ makes with the major-inertia axis $\mathbf{b}_3$ of the body. The projections of the angular momentum toward the reference axis $\mathbf{s}_3$ and the body axis $\mathbf{b}_3$ are $H = G\cos I$ and $L = \cos J$.

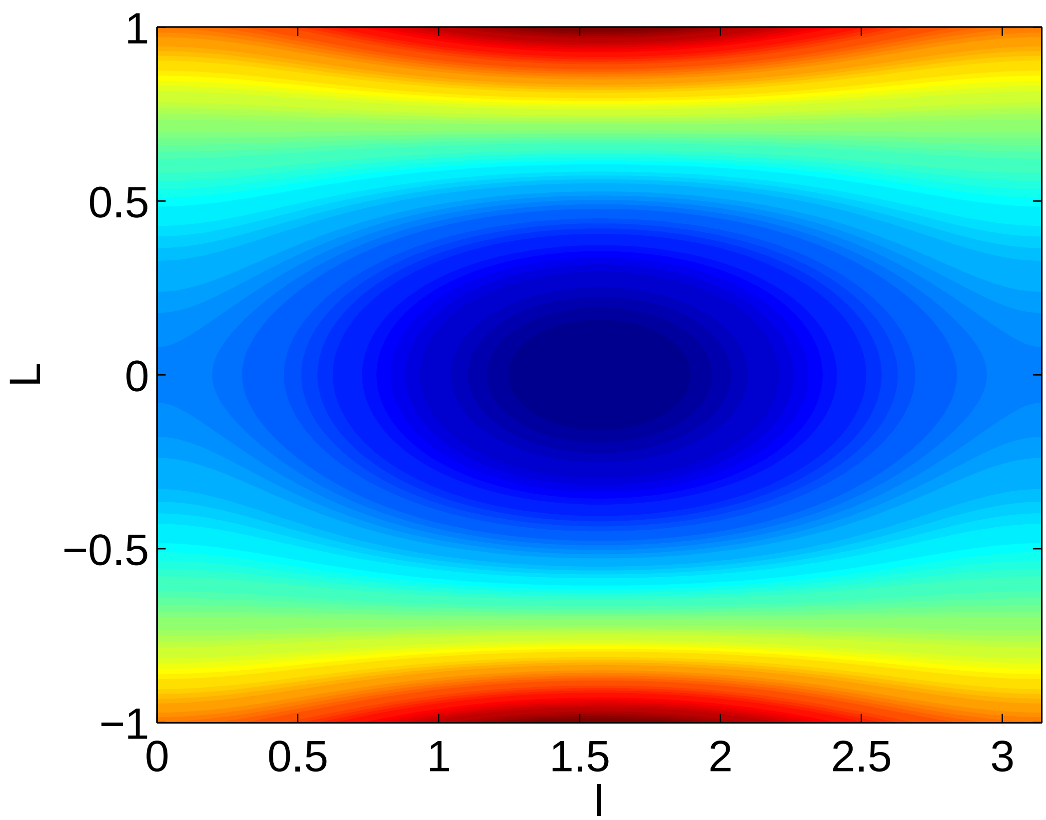

Figure 3: The phase plane of (l, L) comprising the isoenergetic curves of the Serret-Andoyer free-motion Hamiltonian.

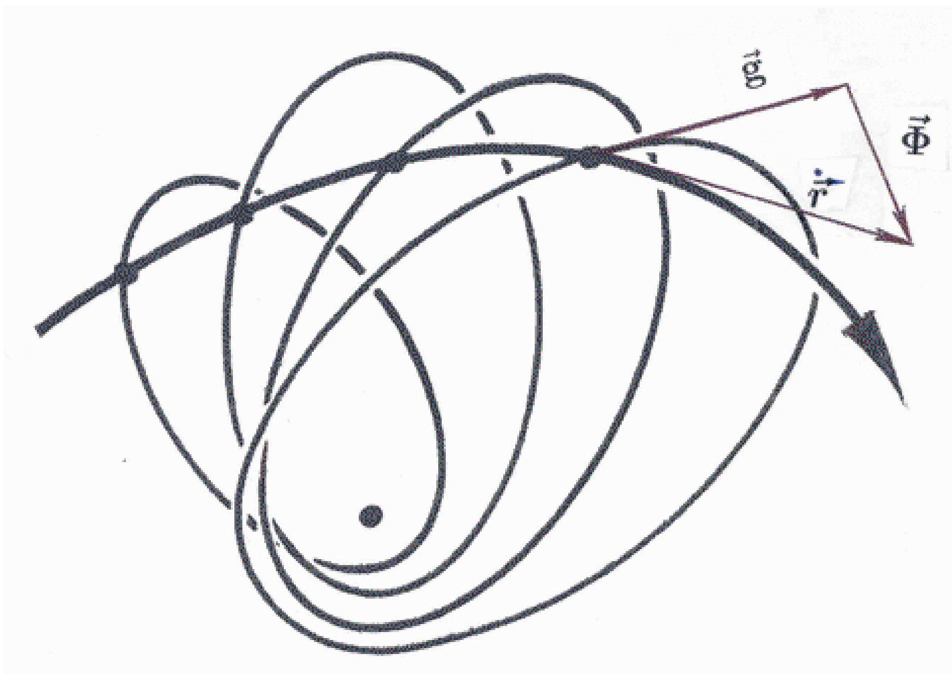

Figure 4: The perturbed trajectory is a set of points belonging to a sequence of confocal instantaneous ellipses. The ellipses are **not** supposed to be tangent, nor even coplanar to the orbit at the intersection point. As a result, the physical velocity $\dot{\boldsymbol{r}}$ (tangent to the trajectory at each of its points) differs from the Keplerian velocity $\mathbf{g}$ (tangent to the ellipse).

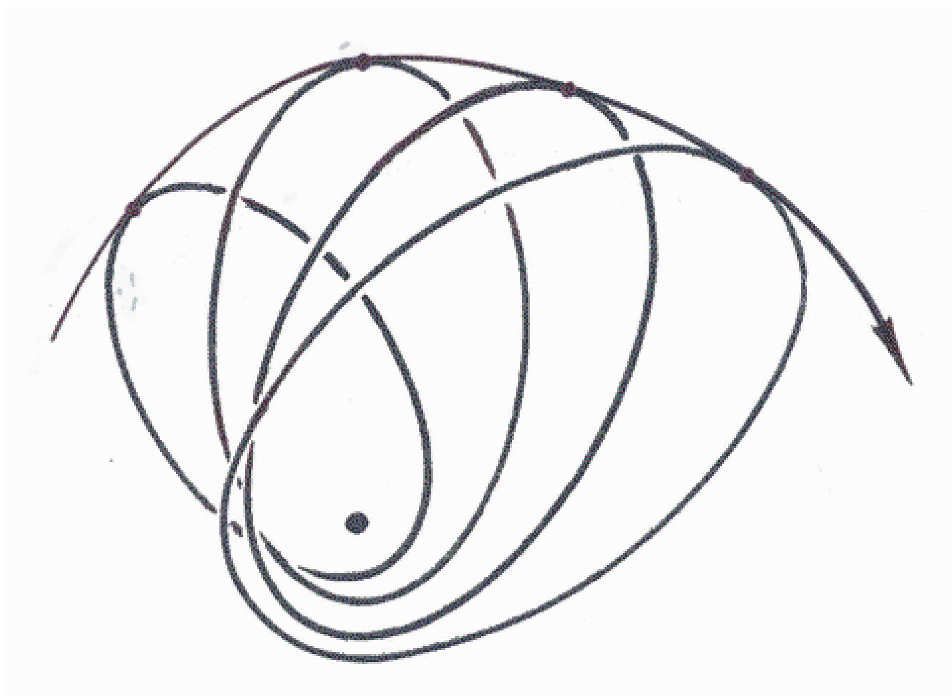

Figure 5: The perturbed trajectory is represented through a sequence of confocal instantaneous ellipses which are tangent to the trajectory at the intersection points, i.e., are osculating. Now, the physical velocity $\dot{\boldsymbol{r}}$ (which is tangent to the trajectory) will coincide with the Keplerian velocity $\mathbf{g}$ (which is tangent to the ellipse)